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A Method for the Generation of a Class of Ill-conditioned Matrices

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Abstract—This paper shows a generalization of Rump's method, which generates a class of matrices with extremely large condition number.

1. Introduction

S. Rump proposed a method to generate a class of *extremely ill-conditioned matrices* [1]. Here the *ill-conditiond* matrix implies those having its condition number such as $10^{50} \sim 10^{100}$ or larger in double precision arithmetic. These matrices are very useful to examine the quality of *accuracy-guaranteed algorithms* for solving linear simultaneous equations [2]–[4]. For this purpose we desire more variety of extremely ill-conditiond matrices.

Rump used a *tricky* technique (shown in Sec. 2.1) to generate an ill-conditioned matrix A. One of key points of his method is to generate a 2×2 matrix V whose determinant is one and whose elements are extrelemy large. To generate V Rump utilized a very special equation, *the PELL EQUATION*. Since his method includes some freedom, we have a class of ill-conditioned matrices of an arbitrary size. *However the class of matrices is not so wide and therefore it is still desirable to generate more variety of ill-conditioned matrices.* This is the motivation of this reseach and this paper studies on *the following three items.*

Item 1. We show that almost the same discussion as Rump's is possible when we use the matrix V generated by the Euclid algorithm instead of the Pell Equation. This extends the Rump's matrices considerably. *Instead of his tricky and rather complicated calculation*, we use *a simple matrix manipulation,* by which the following Item 2 can easily be derived.

Item 2. We show an extension to the case where the matrix V is a 3×3 matrix.

Item 3. Related to Items 1 and 2 above, we show that we can find a 3×3 integer matrix $A' = [a'_{ij}]$ with $|a'_{ij}| < \mu$ and with $|A'| = 1$ by bordering a 2×2 integer matrix.

2. A generalization of Rump's method by means of simple matrix manipulation (Item 1)

2.1. Three key steps in Rump's method

Rump's method is composed of three key steps as:

1. Generation of a 2×2 *integer matrix* V such that
$$V = \begin{bmatrix} P & F \\ Q & G \end{bmatrix}, \quad |V| = \begin{vmatrix} P & F \\ Q & G \end{vmatrix} = 1 \tag{1}$$
where P, Q, F and G are *extremely large positive integers* such as 10^{50}.

2. P, Q, F and G are particularly chosen as
$$F = kQ, \quad G = P \tag{2}$$
and P and $Q > 0$ satisfy *the PELL Equation:*
$$P^2 - kQ^2 = 1 \quad \Rightarrow \quad V = \begin{bmatrix} P & kQ \\ Q & P \end{bmatrix}, \quad |V| = 1 \tag{3}$$

3. P, Q, F and G are realized by decomposing it into relatively small integers and by using the matrix similar to the *companion matrix* as shown later.

As the result he could give an explicit form of the inverse matrix A^{-1} as well as the condition number of A.

2.2. Formulation of the problem

In a similar way as Rump's method we consider:

Problem 1: Generate a class of *integer matrices*, say $B = [b_{ij}]$ satisfying
$$|B| = \pm 1 \tag{4}$$
$$|b_{ij}| < \sigma' \tag{5}$$
where σ' is a number such as 10^8 (for single precision), $10^{16}, 2^{53}$ (for double precision), but may possibly be 2, 10 or 1000 depending on the applications.

2.3. Generalization of Rump's method

Instead of the Pell equation we use *the Euclid algorithm* to determine F and G in Eq.(1) for the prescribed P and Q. As is well-known, this is possible if P and Q have no common factor[5]. For example, we choose P and Q as:
$$P = 2^k, \ Q = 3^m \quad \text{or} \tag{6}$$
$$P = 2^{k1}5^{k2}11^{k3}, \ Q = 3^{m1}7^{m2} \tag{7}$$

Let σ be a large number such as 10^8, 10^{16} and 2^{53}. The *extremely large numbers* P, Q, F and G can be expanded as:

$$
\begin{aligned}
P &= p_n\sigma^n + p_{n-1}\sigma^{n-1} + p_{n-2}\sigma^{n-2} + \cdots + p_1\sigma + p_0 \\
Q &= q_n\sigma^n + q_{n-1}\sigma^{n-1} + q_{n-2}\sigma^{n-2} + \cdots + q_1\sigma + q_0 \\
F &= f_n\sigma^n + f_{n-1}\sigma^{n-1} + f_{n-2}\sigma^{n-2} + \cdots + f_1\sigma + f_0 \\
G &= g_n\sigma^n + g_{n-1}\sigma^{n-1} + g_{n-2}\sigma^{n-2} + \cdots + g_1\sigma + g_0 \\
& \quad 0 < P, Q, F, G < \sigma^{n+1}, \quad 0 \le p_i,\ q_i,\ f_i,\ g_i < \sigma
\end{aligned}
$$

Using the above, we define a $(2n+2) \times (2n+2)$ matrix, A as

$$
A = \left[\begin{array}{ccccccc}
p_n & p_{n-1} & p_{n-2} & p_{n-3} & \cdots & p_1 & p_0 \\
q_n & q_{n-1} & q_{n-2} & q_{n-3} & \cdots & q_1 & q_0 \\
\hline
1 & -\sigma & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & -\sigma & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & -\sigma & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & \ddots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & 1 & -\sigma \\
\hline
0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0
\end{array}\right.
$$

$$
\left.\begin{array}{ccccccc}
f_n & f_{n-1} & f_{n-2} & f_{n-3} & \cdots & f_1 & f_0 \\
g_n & g_{n-1} & g_{n-2} & g_{n-3} & \cdots & g_1 & g_0 \\
\hline
0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\hline
1 & -\sigma & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & -\sigma & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & -\sigma & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & \ddots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & 1 & -\sigma
\end{array}\right] \tag{8}
$$

2.4. Calculation of the inverse matrix A^{-1}

We will calculate A^{-1} for (10). Let

$$
\Gamma \equiv H \oplus H, \quad H \equiv \begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & \sigma^n \\
0 & 1 & 0 & 0 & \cdots & 0 & \sigma^{n-1} \\
0 & 0 & 1 & 0 & \cdots & 0 & \sigma^{n-2} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & 1 & \sigma \\
0 & 0 & 0 & 0 & \cdots & 0 & 1
\end{bmatrix}
$$

Then we have:

$$
A' \equiv A\Gamma \quad \Rightarrow A = A'\Gamma^{-1} \quad \Rightarrow A^{-1} = \Gamma(A')^{-1}
$$

Let p, q, f, g be row vectors of order n and Σ_n^* be $n \times n$ matrix as follows:

$$
\begin{aligned}
p &= [p_n, p_{n-1}, \cdots, p_1], \quad q = [q_n, q_{n-1}, \cdots, q_1] \\
f &= [f_n, f_{n-1}, \cdots, f_1], \quad g = [g_n, g_{n-1}, \cdots, g_1]
\end{aligned}
$$

$$
\Sigma_n^* \equiv \begin{bmatrix}
1 & -\sigma & 0 & \cdots & 0 \\
0 & 1 & -\sigma & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & -\sigma \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix}
$$

and let O_{ml} be $m \times l$ matrix with all entries 0.

Then A' can be rewritten as:

$$
A' = \left[\begin{array}{cc|cc}
p & P & f & F \\
q & Q & g & G \\
\Sigma_n^* & O_{n1} & O_{n,n} & O_{n1} \\
O_{n,n} & O_{n1} & \Sigma_n^* & O_{n1}
\end{array}\right] \tag{9}
$$

Let the $(2n+2)\times(2n+2)$ permutation matrix P_{erm} changing the order of the columns of A' such that

$$
A'' \equiv A'P_{erm} = \left[\begin{array}{cc|cc}
p & f & P & F \\
q & g & Q & G \\
\Sigma_n^* & O_{n,n} & O_{n1} & O_{n1} \\
O_{n,n} & \Sigma_n^* & O_{n1} & O_{n1}
\end{array}\right] \tag{10}
$$

Then we have:

$$
A'' \equiv A'P_{erm} \quad \Rightarrow \quad A' = A''P_{erm}^{-1} \quad \Rightarrow \quad (A')^{-1} = P_{erm}(A'')^{-1} \tag{11}
$$

Let A'' be rewritten as:

$$
A'' = \begin{bmatrix} U & V \\ W & 0 \end{bmatrix}
$$
$$
U : 2 \times 2n, \ V : 2 \times 2, \ W : 2n \times 2n \tag{12}
$$

Then we have

$$
(A'')^{-1} = \begin{bmatrix} 0 & W^{-1} \\ V^{-1} & -V^{-1}UW^{-1} \end{bmatrix} \tag{13}
$$

Since $|V| = 1$, we have:

$$
V^{-1} = \begin{bmatrix} G & -F \\ -Q & P \end{bmatrix} \tag{14}
$$

Since W can be written as:

$$
W = \Sigma_* \oplus \Sigma_* \tag{15}
$$

$$
W^{-1} = \Sigma_*^{-1} \oplus \Sigma_*^{-1} \tag{16}
$$

$$
\Sigma_*^{-1} = \begin{bmatrix}
1 & \sigma & \sigma^2 & \sigma^3 & \cdots & \sigma^{n-1} \\
0 & 1 & \sigma & \sigma^2 & \cdots & \sigma^{n-2} \\
0 & 0 & 1 & \sigma & \cdots & \sigma^{n-3} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 1 & \sigma \\
0 & 0 & 0 & \cdots & 0 & 1
\end{bmatrix}
$$

We can easily calculate $-V^{-1}UW^{-1}$. So we have

$$
A^{-1} = \Gamma P_{erm}(A'')^{-1} = \Gamma P_{erm} \begin{bmatrix} 0 & W^{-1} \\ V^{-1} & -V^{-1}UW^{-1} \end{bmatrix} \tag{17}
$$

of which first three columns are as follows:

$$
\left[\begin{array}{c|c|c|c}
\sigma^n G & -\sigma^n F & 1 + \sigma^n(-G\tilde{p}_n + F\tilde{q}_n) & \cdots \\
\sigma^{n-1} G & -\sigma^{n-1} F & 0 + \sigma^{n-1}(-G\tilde{p}_n + F\tilde{q}_n) & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\hline
-\sigma^n Q & \sigma^n P & \sigma^n(-G\tilde{f}_n + F\tilde{g}_n) & \cdots \\
-\sigma^{n-1} Q & \sigma^{n-1} P & \sigma^{n-1}(-G\tilde{f}_n + F\tilde{g}_n) & \cdots \\
\cdots & \cdots & \cdots & \cdots
\end{array}\right] \tag{18}
$$

2.5. Infinity-norm condition number of A

From Eqs.(18) and (10) we have

$$\begin{aligned}
\|A^{-1}\|_\infty &> \max[\sigma^n(G+F), \sigma^n(Q+P)] \\
\|A\|_\infty &> \max[\sum_0^n\{p_i+f_i\}, \sum_0^n\{q_i+g_i\}] \\
\sigma^n\sum_0^n p_i &\geq P,\ \sigma^n\sum_0^n q_i \geq Q,\ \sigma^n\sum_0^n f_i \geq F,\ \sigma^n\sum_0^n g_i \geq G,
\end{aligned}$$

$$\|A\|_\infty \cdot \|A^{-1}\|_\infty > \max[(P+F),(Q+G)] \cdot \max[(G+F),(Q+P)]$$

Assume without loss of generality that

$$P > Q,\ \ P > F \ \ \Rightarrow \ \ F > G$$

Then we have a final result (Proof is omitted):

$$\|A\|_\infty \cdot \|A^{-1}\|_\infty > (P+F)(Q+P) \approx O(\sigma^{2(n+1)})$$

This is a generalization of Rump's result.

3. Extension to Case where V is a 3×3 matrix (Item 2)

3.1. Outline of extension

Let $P, Q, R, F, G, H, L, M, N$ be extremely large integers with the magnitude less than σ^{n+1} and let

$$V = \begin{bmatrix} P & F & L \\ Q & G & M \\ R & H & N \end{bmatrix}, \quad |V| = 1$$

$$P = p_n\sigma^n + p_{n-1}\sigma^{n-1} + \cdots + p_0, \ \ |p_i| < \sigma \ \text{ etc.},$$

Let A be a $3(n+1)\times 3(n+1)$ matrix such that

$$A = \begin{bmatrix} \tilde{p} & \tilde{f} & \tilde{l} \\ \tilde{q} & \tilde{g} & \tilde{m} \\ \tilde{r} & \tilde{h} & \tilde{n} \\ \Sigma & 0 & 0 \\ 0 & \Sigma & 0 \\ 0 & 0 & \Sigma \end{bmatrix}$$

$\tilde{p}, \tilde{q}, \tilde{r}, \tilde{f}, \ldots, \tilde{n}$: row vectors of order $n+1$

i,e., $\tilde{p} = [p_n, p_{n-1}, \cdots, p_0]$, etc.

Let Σ be an $n\times(n+1)$ matrix such that

$$\Sigma_n = \begin{bmatrix} 1 & -\sigma & 0 & 0 & \cdots & 0 \\ 0 & 1 & -\sigma & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 1 & -\sigma \end{bmatrix} \tag{19}$$

A^{-1} can similarly calculated as previously. Most different point is the calculation of V^{-1}:

$$V^{-1} = \begin{bmatrix} w_{11} & w_{21} & w_{31} \\ w_{12} & w_{22} & w_{32} \\ w_{13} & w_{23} & w_{33} \end{bmatrix}$$

where w_{ij} is a *cofactor* of the (i, j) element of V.

3.2. Infinity-norm condition number of A

Infinity-norm condition number of A can be calculated as follows:

$$\begin{aligned}
\|A^{-1}\|_\infty > \ & \max[\sigma^n(|w_{11}|+|w_{21}|+|w_{31}|), \\
& \sigma^n(|w_{12}|+|w_{22}|+|w_{32}|), \\
& \sigma^n(|w_{11}|+|w_{21}|+|w_{31}|)]
\end{aligned}$$

$$\begin{aligned}
\|A\|_\infty > \max\Bigg[& \sum_0^n\{\tilde{p}_i+\tilde{f}_i+\tilde{l}_i\},\ \sum_0^n\{\tilde{q}_i+\tilde{g}_i+\tilde{m}_i\}, \\
& \sum_0^n\{\tilde{r}_i+\tilde{h}_i+\tilde{n}_i\}\Bigg]
\end{aligned}$$

Using the formula:

$$|V|V\begin{pmatrix} i & j \\ k & l \end{pmatrix} = w_{ik}w_{jl} - w_{il}w_{jk}$$

we can derive

$$P + F + L = \frac{N}{L}(P+Q+R), \quad \text{etc.},$$

Without loss of generality we assume

$$P, Q, F, G, \text{and } w_{33} \text{ are very large.}$$

Then we have the final result:

$$\|A\|_\infty \cdot \|A^{-1}\|_\infty > (P+F+L)\{|w_{31}|+|w_{32}|+|w_{33}|\} \approx O(\sigma^{3(n+1)})$$

4. Generation of a third order integer matrix with the determinant one

. **Problem 2**: Find a 3×3 integer matrix $A' = [a'_{ij}]$ by bordering the prescribed 2×2 matrix $A = [a_{ij}]$. Here $|a'_{ij}| < \mu$, where μ corresponds to σ^{n+1} in the previous sections.

$$A' = \begin{bmatrix} a_{11} & a_{12} & y_1 \\ a_{21} & a_{22} & y_2 \\ x_1 & x_2 & z \end{bmatrix}, \quad |A'| = 1 \tag{20}$$

$$|a_{ij}| < \mu, \ \ |x_i| < \mu, \ \ |y_i| < \mu, \ \ |z| < \mu \tag{21}$$

Of course we have to impose some restrictions on a_{ij}. **Theorem 1:** An $n\times n$ integer matrix A can be bordered so that its determinant is equal to one, *only if* the $(n-1)$th determinant (common) divisor is one.

In the case of $n = 2$ this means that there is no common factor among all a_{ij} $(i, j = 1, 2)$. As a special case we impose:

Assumption 1: a_{21} and a_{22} have no common factor.

Theorem 2: On Assumption 1 we can find x_1, x_2, y_1, y_2 and z satisfying the conditions in Problem 2.

Proof of Theorem 2)

Lemma 1: By the *Euclid algorithm* we can choose x_1 and x_2 such that

$$\begin{vmatrix} a_{21} & a_{22} \\ x_1 & x_2 \end{vmatrix} = 1, \ \ |x_i| < \mu$$

Then $|A'|$ can be written as:

$$
\begin{aligned}
|A'| &= y_1 \begin{vmatrix} a_{21} & a_{22} \\ x_1 & x_2 \end{vmatrix} - y_2 \begin{vmatrix} a_{11} & a_{12} \\ x_1 & x_2 \end{vmatrix} + z \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\
&\equiv y_1 + ay_2 + bz
\end{aligned}
$$

Here in general

$$|a| < \mu^2, \quad |b| < \mu^2$$

Lemma 2: Let a and b are integers satisfying

$$|a|, |b| < \mu^2.$$

Then there exist y_2 and z satisfying $|y_2|, |z| < \mu$ and

$$|ay_2 + bz| < \mu.$$

We can prove Lemma 2 by using Euclid algorithm.
Using Lemma 2, we can choose y_1 such that

$$|y_1 + ay_2 + bz| = 1, |y_1| < \mu$$

Proof of Lemma 2) Assume that $a > b > 0$ and let $x_0 \equiv a$ and $x_1 \equiv b$. The Euclid algorithm derives the following:

$$
\begin{aligned}
x_0 &= k_0x_1 + x_2 \ \ (0 < x_2 < x_1) \\
x_1 &= k_1x_2 + x_3 \ \ (0 < x_3 < x_2) \\
x_2 &= k_2x_3 + x_4 \ \ (0 < x_4 < x_3) \\
&\vdots \\
x_{n-1} &= k_{n-1}x_n + x_{n+1} \ \ (0 < x_{n+1} < x_n) \\
x_n &= k_nx_{n+1} + x_{n+2} \ \ (0 < x_{n+2} < x_{n+1})
\end{aligned}
$$

From this we have the following expression:

$$
\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix}
$$
$$p_0 = 1, \ \ p_1 = k_0, \ \ p_n = p_{n-1}k_{n-1} + p_{n-2}$$
$$q_0 = 0, \ \ q_1 = 1, \ \ q_n = q_{n-1}k_{n-1} + q_{n-2}$$

The following important formula holds:

$$\begin{vmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{vmatrix} = (-1)^n$$

We therefore have:

$$
\begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} = (-1)^n \begin{bmatrix} q_{n-1} & -p_{n-1} \\ -q_n & p_n \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}
$$

or

$$
\begin{aligned}
x_n &= (-1)^n[q_{n-1}x_0 - p_{n-1}x_1] \\
x_{n+1} &= (-1)^n[-q_nx_0 + p_nx_1]
\end{aligned}
$$

Lemma 2 can be proved by showing $0 < p_n < \mu$ from the following theorem:

Theorem 3: Assume that

$$\mu^2 > x_0 > x_1 > \mu > 0 \tag{22}$$

and let n be an integer such that

$$x_n > \mu, \quad x_{n+1} < \mu \tag{23}$$

Then

$$\mu > p_n > q_n (> 0) \tag{24}$$

Proof of Theorem 3) Assume first that n is even. Then the above conditions means

$$
\begin{aligned}
q_{n-1}x_0 - p_{n-1}x_1 &> \mu \\
0 < -q_nx_0 + p_nx_1 &< \mu
\end{aligned}
$$

Since $p_nq_{n-1} - p_{n-1}q_n = 1$ holds, we have:

$$q_{n-1} = \frac{1 + p_{n-1}q_n}{p_n}$$

which is substituted into the above, we have:

$$\frac{1 + p_{n-1}q_n}{p_n} x_0 - p_{n-1}x_1 > \mu$$

from which we have

$$x_0 + p_{n-1}(q_nx_0 - p_nx_1) > p_n\mu$$

Since $\mu^2 > x_0$ and $0 > q_nx_0 - p_nx_1 > -\mu$ hold, we have:

$$\mu^2 > p_n\mu, \ \ \text{i.e.,} \ \ p_n < \mu$$

This completes the proof for n even. The case of n odd can be treated similarly.

5. Conclusion

We show some generalization of Rump's method to generate extremely ill-condition matrices.

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