2009 International Symposium on Nonlinear Theory and its Applications
NOLTA'09, Sapporo, Japan, October 18-21, 2009

From Chaotic Attractors to Complex Networks

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Abstract—Recently, novel approaches have been proposed to analyze nonlinear time series using the complex network theory. In these approaches, time series are transformed to networks and analyzed by the complex network theory. We have already proposed a recurrence-plot based method to construct networks from attractors of nonlinear dynamical systems or reconstructed attractors from time series and showed that the networks constructed from chaotic attractors become the small-world networks. Here we show that the small-world property emerges in several chaotic dynamical systems. We also investigate the influence of dynamical noise injected to the dynamical system on networks constructed from attractors generated from these dynamical systems.

1. Introduction

Recent studies based on the analyses of nonlinear dynamical systems using the complex network theory have reported various relationships between nonlinear dynamics and complex networks [1–4]. In these methods, the networks are constructed from time series and the constructed networks are analyzed through the complex network theory. The construction methods of networks are classified into two types based on whether the time series is analyzed in the temporal domain or not. For example, in Ref. [1], a pseudo-periodic time series is divided by its extrema to determine its periods and the divided time series is recognized as a node in a network. The method in Ref. [2] also directly transforms a time series waveform to a network by using the visibility relation between two values in the time series. Although these methods can reflect the dynamical behavior of the time series, it is not so easy to capture the topological features of the dynamics which produces complecated time series. In this sense, the construction method using the recurrence plot can be an effective and important tool to reveal the topological structures of nonlinear dynamics [3–5] because these methods analyze the networks constructed from attractors reconstructed from time series based on the embedding theorem [6]. Although the construction methods in Refs. [3, 4] are essentially the same, the strategy for analyzing the constructed networks are different. In Ref. [4], the networks constructed from various types of the attractors are grouped by the rank of the network motif [8]. On the other hand, in Ref. [3], basic statistics such as the clustering coefficient and the characteristic path length are used to analyze the networks constructed from attractors through the bifurcation phenomenon of the nonlinear dynamical system, (the Rössler system is used in Ref. [3]), and showed that the networks constructed from chaotic attractors have the small-world property. In Ref. [5], it is also shown that fit-get-rich networks are emerged from chaotic attractors.

In this paper, we show that the emergence of the small-world property from the chaotic attractors is universal property observed in the constructed networks. We further investigate the structural features of the networks constructed from the attractors under more realistic conditions in which the dynamical noise is injected to the dynamical system.

2. Method

We first construct a network from an attractor. A method for constructing networks is the same as generating the recurrence plot [9]. Let us describe $\boldsymbol{x}(i)$ ($i = 1, \ldots, N$) the ith point on the attractor produced from a dynamical system. The distance between $\boldsymbol{x}(i)$ and $\boldsymbol{x}(j)$ is defined by $d_{ij} = |\boldsymbol{x}(i) - \boldsymbol{x}(j)|$. Here, let r_{ij} be the $(i,\ j)$th entry of a two-dimensional $N \times N$ square matrix R. The algorithm for generating the recurrence plots is described as follows:

1. Select the ith point $\boldsymbol{x}(i)$ and calculate d_{ij} for all j.
2. Select M nearest neighbor points of $\boldsymbol{x}(i)$. These points are described as $\boldsymbol{x}(k_1), \ldots, \boldsymbol{x}(k_M)$.
3. Set $r_{ij} = 1$ if $j \in I_i$ and $r_{ij} = 0$ if $j \notin I_i$ where I_i is a set of indices of the selected M nearest neighbor points $\{k_1, \ldots, k_M\}$.
4. Repeat steps 1 to 3 for all i.
5. Symmetrize the matrix R. Namely, for all pairs i and j, if $r_{ij} = 1$, then $r_{ji} = 1$.

We regard this matrix R as an adjacency matrix of the network. Next, we evaluate the network using two basic measures of the complex network theory, clustering coefficient C and characteristic path length L [3].

2.1. Measures of the complex network theory

The clustering coefficient and the characteristic path length [10] are fundamental measures to evaluate the characteristic properties in the complex network theory [10]. In particular, the small world property of the networks can be evaluated by the clustering coefficient and the characteristic path length.

The clustering coefficient is defined as follows:

$$C = \frac{1}{N}\sum_{i=1}^{N} C_i, \quad C_i = \frac{l_i}{{}_{k_i}\mathrm{C}_2}, \tag{1}$$

where k_i is the number of adjacent nodes of the ith node, l_i is the number of connections between the adjacent nodes of the ith node, and N is the number of nodes. The clustering coefficient shows a local connectivity among any three nodes in the network.

The characteristic path length is defined as follows:

$$L = \frac{1}{N(N-1)}\sum_{i=1}^{N}\sum_{j=1, i\neq j}^{N} d_{ij}, \tag{2}$$

where d_{ij} is the shortest path length between the ith and the jth nodes. The characteristic path length shows global accessibility of the network.

If a network has a high clustering coefficient and small characteristic path length, the network has the small world property. Such a network is called a small world network.

3. Chaotic attractors have small-world property

To investigate the universality of emergence of the small-world property from the chaotic attractors, we introduce three mathematical models: the Rössler system [11], the Lorenz system [12], and the Chua circuit [13] and conducted the same experiments as Ref. [3]. The Rössler system is described by

$$\begin{cases} \dot{x} = -(y+z), \\ \dot{y} = x + 0.2y, \\ \dot{z} = 0.2 + z(x-c). \end{cases} \tag{3}$$

The Lorenz system is described by

$$\begin{cases} \dot{x} = -10x + 10y, \\ \dot{y} = -xz + 28x - y, \\ \dot{z} = xy - bz. \end{cases} \tag{4}$$

The Chua circuit is described by

$$\begin{cases} \dot{x_1} = \alpha(x_2 - h(x_1)), \\ \dot{x_2} = x_1 - x_2 + x_3, \\ \dot{x_3} = -\frac{100}{7^3}x_2. \end{cases} \tag{5}$$

where

$$h(x_1) = \begin{cases} -\frac{5}{7}x_1 + -\frac{3}{7}, & x_1 \leq 1 \\ -\frac{8}{7}x_1, & |x_1| \geq 1 \\ -\frac{5}{7}x_1 + \frac{3}{7}, & x_1 \geq -1. \end{cases} \tag{6}$$

We investigate characteristic structural properties of the constructed networks through the bifurcation structure: we vary the parameter c in Eq. (3), b in Eq. (4), and α in Eq. (5). For each parameter, we calculate C and L of the constructed networks from the attractors. Throughout this paper, we set the number of nodes $N = 12,000$, the number of adjacent nodes $M = 20$, and the time step of the numerical integration $\delta t = 0.01$. Both C and L are normalized by the largest value.

We first analyze the networks constructed from the Chua circuit with network size N. In Fig. 1(a), the clustering coefficient C of the periodic attractors shows a similar tendency if the number of nodes N increases, and the chaotic attractor also takes similar values of C to the periodic ones. In addition, in Fig. 1(b), although the periodic attractors have large characteristic path length L but the chaotic ones keep small values of L when the number of nodes N increases.

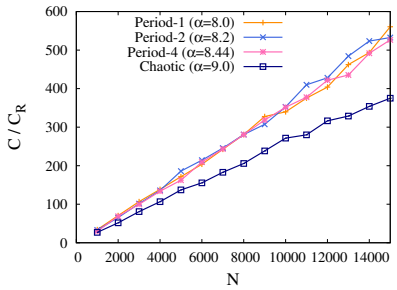
From Figs. 1(a) and (b), the networks constructed from the chaotic attractors gradually converged to a small world network when the size of networks N becomes large.

Figure 2 shows the results of normalized values of the clustering coefficients C^* and the characteristic path length L^* with the bifurcation diagrams of (a) the Rössler system, (b) the Lorenz system, and (c) the Chua circuit. The upper figures of Figs. 2 (a), (b), and (c) are the bifurcation diagrams. In each figure, to generate the bifurcation diagrams, we extracted the local maxima of the first variables of Eqs. (3), (4), and (5), and the nth maximum is plotted. From Fig. 2, we can see that both C^* and L^* take large values for the periodic attractors. On the other hand, in the chaotic regions, C^* becomes large but L^* becomes small. Then, we can confirm that the chaotic attractors generated from these mathematical models have the small-world properties.

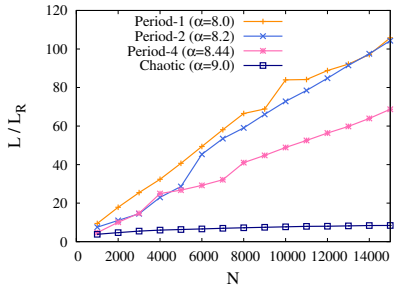
4. The small-world property and dynamical noise

To investigate how the dynamical noise influences the small-world property of the networks constructed from the chaotic attractors, we conducted the same experiments with the following noisy Rössler system:

$$\begin{cases} \dot{x} = -(y+z) + \eta_x(\sigma), \\ \dot{y} = x + 0.2y + \eta_y(\sigma), \\ \dot{z} = 0.2 + z(x-c) + \eta_z(\sigma), \end{cases} \tag{7}$$



(a) clustering coefficients



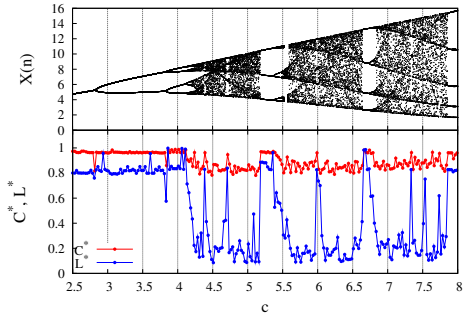
(b) characteristic path lengths

Figure 1: Results of (a) the clustering coefficients C and (b) the characteristic path lengths L in case of increasing N for the Chua circuit. In these results, the number of adjacent nodes is fixed: $M = 20$.

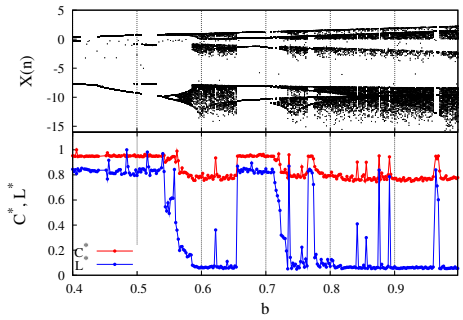
where $\eta_x(\sigma)$, $\eta_y(\sigma)$, and $\eta_z(\sigma)$ are dynamical noises whose average and variance are zero and σ. To evaluate structural properties of the constructed networks through the bifurcation, we varied the parameter c in Eq. (7).

Figure 3 shows the results of C^* and L^* with the bifurcation diagram for the noisy Rössler system (Eq. (7)). The upper figures of Fig. 3 show the bifurcation diagrams. Figures 3 (a), (b), and (c) show the results of the three cases of $\sigma = 0.1$, 0.4, and 0.9, respectively. From the bifurcation diagram in Fig. 3 (b), we can see that the original bifurcation structures are destroyed. In particular, periodic windows disappear when the dynamical noise is added. Although the periodic window is invisible, the results of L^* show remarkable changes in the regions where the periodic window exist in the original diagram; for example $c = 4.4$, 4.7, and 6.0. If the amount of the dynamical noise increases, these changes of L^* become small. In addition, C^* and L^* gradually take small values if the dynamical noise increases.

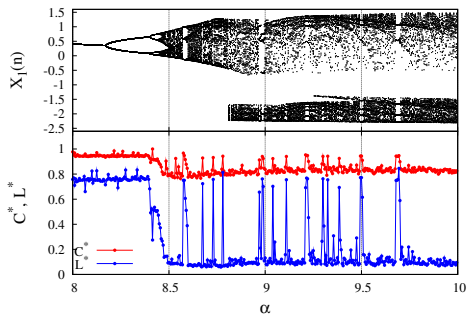
From Fig. 3, we can also confirm that L^* tends to decrease when the parameter c approaches bifurcation points. The decrease of L^* is generally caused by increase of short-cuts in the network. The results of Figure 3 indicate that the number of the short-cuts in the networks constructed from the attractors in-



(a) The Rössler system



(b) The Lorenz system



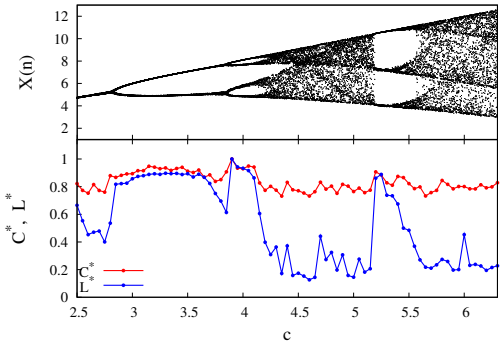
(c) The Chua circuit

Figure 2: The results of the clustering coefficients C^* and the characteristic path lengths L^* for (a) the Rössler system, (b) the Lorenz system, and (c) the Chua circuit. The upper figures are bifurcation diagrams of each system.

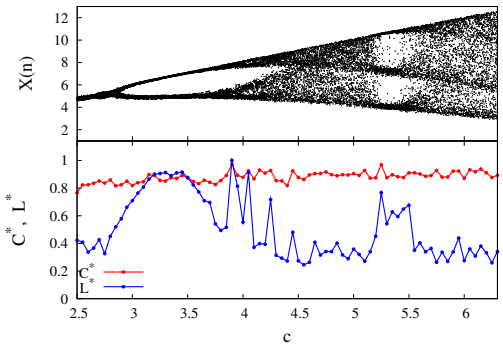
creases when the parameter value moves close to the bifurcation points. Then, it is considered that the networks constructed from the periodic attractors become sensitive to the dynamical noise as the parameter c approaches the bifurcation points and those networks reflect this sensitivity by the increase of short-cuts.

5. Conclusion

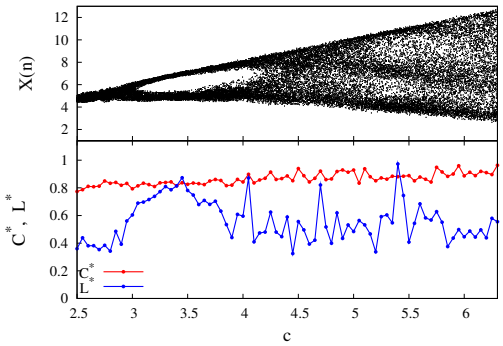
In this paper, we used several mathematical models of nonlinear dynamical systems and evaluated whether the networks constructed from the chaotic attractors have the small-world property or not. As a result,



(a) $\sigma = 0.1$



(b) $\sigma = 0.4$



(c) $\sigma = 0.9$

Figure 3: Averaged values of the clustering coefficients C^* and the characteristic path length L^* for 10 simulations.

we showed that the chaotic attractors have the small-world property. In the experiments using the noisy Rössler system, we showed that the networks constructed from the chaotic attractors lose the small-world property. However, we also find that the dynamical noise injected to the dynamical system increases short-cuts in the networks constructed from attractors.

To elucidate the relationship between the chaotic attractors and complex networks, one of the important strategies is to involve the information about temporal evolutions of attractors into the proposed method and evaluate the constructed networks through the evolution processes of the networks [5].

The research of TI is partially supported by a Grant-in-Aid for Exploratory Research (No. 20650032) from JSPS.

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