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Minimization of l_2-Sensitivity for State-Estimate Feedback Controllers Subject to l_2-Scaling Constraints Using Quasi-Newton Method

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Abstract—The l_2-sensitivity of a closed-loop transfer function with respect to coefficients of a state-estimate feedback controller is analyzed, and the problem of minimizing an l_2-sensitivity measure subject to l_2-scaling constraints is formulated. Next, the constraint optimization problem is converted into an unconstrained optimization formulation by using linear-algebraic techniques, and an efficient quasi-Newton method is applied to solve the unconstrained optimization problem. Finally, a numerical example is presented to illustrate the utility of the proposed technique.

1. Introduction

It is well-known that a linear system has an infinite number of state-space minimal realizations. For a given transfer function, it is of practical importance to construct a state-space realization such that the coefficient sensitivity of the linear system is minimal or nearly minimal in a certain sense. Due to finite word length (FWL) effects caused by either truncation or rounding of the coefficients, the poor sensitivity may lead to the degradation of the transfer characteristics in a FWL implementation of the system. Several techniques for constructing state-space realizations with minimum sensitivity have been reported: l_1/l_2-mixed sensitivity minimization [1]-[4] and l_2-sensitivity minimization [5]-[8]. It has been argued in [9],[10] that sensitivity minimization based on a pure l_2-norm is more natural and reasonable relative to l_1/l_2-mixed sensitivity minimization. More recently, the minimization problem of l_2-sensitivity subject to l_2-scaling constraints has been explored for state-space digital filters [9],[10]. However, not enough research has been done on the minimization of either l_1/l_2-mixed sensitivity or l_2-sensitivity subject to l_2-scaling constraints for the closed-loop transfer function with a state-estmate feedback controller [11]. Notice that the introduction of l_2-scaling constraints is beneficial for suppressing overflow.

In this paper, the problem of synthesizing the optimal structure of a state-estimate feedback digital controller with minimum l_2-sensitivity and no overflow is investigated. First, the l_2-sensitivity of a closed-loop transfer function with respect to coefficients of a state-estimate feedback controller is analyzed. Second, the problem of minimizing the l_2-sensitivity subject to l_2-scaling constraints is formulated. Third, the constrained optimization problem is converted into an unconstrained one by using linear-algebraic techniques. The unconstrained optimization problem is then solved by applying a quasi-Newton algorithm. Finally, a numerical example is presented to illustrate the validity and effectiveness of the proposed technique.

2. l_2-Sensitivity Analysis

Suppose that a linear discrete-time time-invariant system is represented by

$$\begin{aligned} \boldsymbol{x}(k+1) &= \boldsymbol{A}_o\boldsymbol{x}(k) + \boldsymbol{b}_o u(k) \\ y(k) &= \boldsymbol{c}_o\boldsymbol{x}(k) \end{aligned} \tag{1}$$

where $\boldsymbol{x}(k)$ is an $n \times 1$ state vector, $u(k)$ is a scalar input, $y(k)$ is a scalar output, and $\boldsymbol{A}_o$, $\boldsymbol{b}_o$ and $\boldsymbol{c}_o$ are $n \times n$, $n \times 1$ and $1 \times n$ real matrices, respectively. The above linear system is assumed to be stable, controllable and observable. The transfer function of the linear system in (1) is given by

$$H_o(z) = \boldsymbol{c}_o(z\boldsymbol{I}_n - \boldsymbol{A}_o)^{-1}\boldsymbol{b}_o. \tag{2}$$

Assuming that a regulator is designed using the full-order state observer, a state-estimate feedback controller denoted by $(\boldsymbol{D}_o, \boldsymbol{b}_o, \boldsymbol{g}_o, \boldsymbol{k}_o)_n$ can be expressed as

$$\begin{aligned} \tilde{\boldsymbol{x}}(k+1) &= \boldsymbol{D}_o\tilde{\boldsymbol{x}}(k) + \boldsymbol{b}_o u(k) + \boldsymbol{g}_o y(k) \\ u(k) &= -\boldsymbol{k}_o\tilde{\boldsymbol{x}}(k) + r(k) \end{aligned} \tag{3}$$

where $\tilde{\boldsymbol{x}}(k)$ is an $n \times 1$ state vector in the full-order state observer, $\boldsymbol{g}_o$ is an $n \times 1$ gain vector chosen so that all the eigenvalues of matrix $\boldsymbol{D}_o = \boldsymbol{A}_o - \boldsymbol{g}_o\boldsymbol{c}_o$ are located within the unit circle on the complex plane, $\boldsymbol{k}_o$ is a $1 \times n$ state-feedback gain vector chosen so that each of the eigenvalues of matrix $\boldsymbol{A}_o - \boldsymbol{b}_o\boldsymbol{k}_o$ is at an arbitrary location within the unit circle, and $r(k)$ is a scalar reference signal.

In the case where the state-estimate feedback controller in (3) is implemented with infinite precision from the coefficients of the linear system in (1), the FWL implementation of (3) can be written in the form

$$\begin{aligned} \hat{\boldsymbol{x}}(k+1) &= \boldsymbol{D}\hat{\boldsymbol{x}}(k) + \boldsymbol{b}u(k) + \boldsymbol{g}y(k) \\ u(k) &= -\boldsymbol{k}\hat{\boldsymbol{x}}(k) + r(k) \end{aligned} \tag{4}$$

where $\hat{\boldsymbol{x}}(k)$ denotes the state vector in the FWL implementation. The closed-loop system consisting of (1) and (4) is then described by

$$\begin{bmatrix} \boldsymbol{x}(k+1) \\ \hat{\boldsymbol{x}}(k+1) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_o & -\boldsymbol{b}_o\boldsymbol{k} \\ \boldsymbol{g}\boldsymbol{c}_o & \boldsymbol{D}-\boldsymbol{b}\boldsymbol{k} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(k) \\ \hat{\boldsymbol{x}}(k) \end{bmatrix} + \begin{bmatrix} \boldsymbol{b}_o \\ \boldsymbol{b} \end{bmatrix} r(k)$$
$$y(k) = [\boldsymbol{c}_o \quad \boldsymbol{0}] \begin{bmatrix} \boldsymbol{x}(k) \\ \hat{\boldsymbol{x}}(k) \end{bmatrix}. \tag{5}$$

The transfer function of the linear system in (5) is expressed as

$$H_c(z) = \overline{\boldsymbol{c}}\,(z\boldsymbol{I}_{2n} - \overline{\boldsymbol{A}})^{-1}\overline{\boldsymbol{b}} \tag{6}$$

where

$$\overline{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{A}_o & -\boldsymbol{b}_o\boldsymbol{k} \\ \boldsymbol{g}\boldsymbol{c}_o & \boldsymbol{D}-\boldsymbol{b}\boldsymbol{k} \end{bmatrix}, \quad \overline{\boldsymbol{b}} = \begin{bmatrix} \boldsymbol{b}_o \\ \boldsymbol{b} \end{bmatrix}, \quad \overline{\boldsymbol{c}} = [\boldsymbol{c}_o \quad \boldsymbol{0}].$$

Definition 1: Let $\boldsymbol{X}$ be an $m \times n$ real matrix and let $f(\boldsymbol{X})$ be a scalar complex function of $\boldsymbol{X}$, differentiable with respect to all the entries of $\boldsymbol{X}$. The sensitivity function of $f(\boldsymbol{X})$ with respect to $\boldsymbol{X}$ is then defined as

$$\boldsymbol{S}_X = \frac{\partial f(\boldsymbol{X})}{\partial \boldsymbol{X}} \quad \text{with} \quad (\boldsymbol{S}_X)_{ij} = \frac{\partial f(\boldsymbol{X})}{\partial x_{ij}} \tag{7}$$

where x_{ij} denotes the (i,j)th entry of matrix $\boldsymbol{X}$.

According to Definition 1, the sensitivities of $H_c(z)$ with respect to $\boldsymbol{D}$, $\boldsymbol{b}$, $\boldsymbol{k}$ and $\boldsymbol{g}$ are defined and then evaluated with the exact values of $\boldsymbol{A}_o$, $\boldsymbol{b}_o$, $\boldsymbol{c}_o$, $\boldsymbol{k}_o$ and $\boldsymbol{g}_o$ as

$$\begin{aligned} \frac{\partial H_c(z)}{\partial \boldsymbol{D}} &= -[\boldsymbol{F}(z)W_1(z)\boldsymbol{G}(z)]^T \\ \frac{\partial H_c(z)}{\partial \boldsymbol{b}} &= -W_1(z)(1-W_2(z))\boldsymbol{G}^T(z) \\ \frac{\partial H_c(z)}{\partial \boldsymbol{k}^T} &= -\boldsymbol{F}(z)W_1(z), \quad \frac{\partial H_c(z)}{\partial \boldsymbol{g}} = -W_1^2(z)\boldsymbol{G}^T(z) \end{aligned} \tag{8}$$

where

$$\begin{aligned} \boldsymbol{G}(z) &= \boldsymbol{k}_o\left[z\boldsymbol{I}_n - (\boldsymbol{A}_o - \boldsymbol{g}_o\boldsymbol{c}_o)\right]^{-1} \\ \boldsymbol{F}(z) &= \left[z\boldsymbol{I}_n - (\boldsymbol{A}_o - \boldsymbol{b}_o\boldsymbol{k}_o)\right]^{-1}\boldsymbol{b}_o \\ W_1(z) &= \boldsymbol{c}_o\left[z\boldsymbol{I}_n - (\boldsymbol{A}_o - \boldsymbol{b}_o\boldsymbol{k}_o)\right]^{-1}\boldsymbol{b}_o \\ W_2(z) &= \boldsymbol{k}_o\left[z\boldsymbol{I}_n - (\boldsymbol{A}_o - \boldsymbol{b}_o\boldsymbol{k}_o)\right]^{-1}\boldsymbol{b}_o. \end{aligned}$$

Definition 2 [5]: Let $\boldsymbol{X}(z)$ be an $m \times n$ complex matrix-valued function of the complex variable z and let $x_{pq}(z)$ be the (p,q)th entry of $\boldsymbol{X}(z)$. The l_2-norm of $\boldsymbol{X}(z)$ is then defined as

$$\begin{aligned} \|\boldsymbol{X}(z)\|_2 &= \left[\frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{p=1}^{m}\sum_{q=1}^{n}\left|x_{pq}(e^{j\omega})\right|^2\right)d\omega\right]^{\frac{1}{2}} \\ &= \left(\text{tr}\left[\frac{1}{2\pi j}\oint_{|z|=1}\boldsymbol{X}(z)\boldsymbol{X}^*(z)\frac{dz}{z}\right]\right)^{\frac{1}{2}}. \end{aligned}$$

By virtue of (8) and Definition 2, it is possible to define the overall l_2-sensitivity measure by

$$m_2 = \left\|\frac{\partial H_c(z)}{\partial \boldsymbol{D}}\right\|_2^2 + \left\|\frac{\partial H_c(z)}{\partial \boldsymbol{b}}\right\|_2^2 + \left\|\frac{\partial H_c(z)}{\partial \boldsymbol{k}^T}\right\|_2^2 + \left\|\frac{\partial H_c(z)}{\partial \boldsymbol{g}}\right\|_2^2 \tag{9}$$

Note that the sensitivity measure in [4] based on a mixture of l_1- and l_2-norms differs from that in (9). By substituting (8) into (9), we arrive at

$$m_2 = \text{tr}\,[\boldsymbol{M}_1] + \text{tr}\,[\boldsymbol{W}_9] + \text{tr}\,[\boldsymbol{K}_1] + \text{tr}\,[\boldsymbol{N}_9] \tag{10}$$

where $\boldsymbol{M}_1$, $\boldsymbol{W}_9$, $\boldsymbol{K}_1$, and $\boldsymbol{N}_9$ are obtained by a general expression

$$\boldsymbol{Y} = \frac{1}{2\pi j}\oint_{|z|=1}\boldsymbol{X}(z)\boldsymbol{X}^*(z)\frac{dz}{z}$$

with
$\boldsymbol{X}(z) = \boldsymbol{F}(z)W_1(z)\boldsymbol{G}(z)$ for $\boldsymbol{Y} = \boldsymbol{M}_1$
$\boldsymbol{X}(z) = [W_1(z)(1-W_2(z))\boldsymbol{G}(z)]^*$ for $\boldsymbol{Y} = \boldsymbol{W}_9$
$\boldsymbol{X}(z) = \boldsymbol{F}(z)W_1(z)$ for $\boldsymbol{Y} = \boldsymbol{K}_1$
$\boldsymbol{X}(z) = [W_1^2(z)\boldsymbol{G}(z)]^*$ for $\boldsymbol{Y} = \boldsymbol{N}_9$.

Taking the z-transform of the state equation in (5) yields

$$\begin{bmatrix} \boldsymbol{X}(z) \\ \hat{\boldsymbol{X}}(z) \end{bmatrix} = (z\boldsymbol{I}_{2n} - \overline{\boldsymbol{A}})^{-1}\overline{\boldsymbol{b}}R(z). \tag{11}$$

Replacing $\boldsymbol{D}$, $\boldsymbol{b}$, $\boldsymbol{g}$, and $\boldsymbol{k}$ by the exact values of $\boldsymbol{D}_o$, $\boldsymbol{b}_o$, $\boldsymbol{g}_o$, and $\boldsymbol{k}_o$, respectively, one can write (11) as

$$\begin{bmatrix} \boldsymbol{X}(z) \\ \hat{\boldsymbol{X}}(z) \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}(z) \\ \boldsymbol{F}(z) \end{bmatrix} R(z) \tag{12}$$

where

$$\boldsymbol{F}(z) = \boldsymbol{S}(z\boldsymbol{I}_{2n} - \boldsymbol{S}^{-1}\overline{\boldsymbol{A}}_o\boldsymbol{S})^{-1}\boldsymbol{S}^{-1}\overline{\boldsymbol{b}}_o$$
$$\overline{\boldsymbol{A}}_o = \begin{bmatrix} \boldsymbol{A}_o & -\boldsymbol{b}_o\boldsymbol{k}_o \\ \boldsymbol{g}_o\boldsymbol{c}_o & \boldsymbol{D}_o - \boldsymbol{b}_o\boldsymbol{k}_o \end{bmatrix}, \qquad \overline{\boldsymbol{b}}_o = \begin{bmatrix} \boldsymbol{b}_o \\ \boldsymbol{b}_o \end{bmatrix}.$$

Then it is easy to show that the controllability Gramian

$$\boldsymbol{K}_c = \frac{1}{2\pi j}\oint_{|z|=1}\boldsymbol{F}(z)\boldsymbol{F}^*(z)\frac{dz}{z} \tag{13}$$

can be obtained by solving the Lyapunov equation

$$\boldsymbol{K}_c = (\boldsymbol{A}_o - \boldsymbol{b}_o\boldsymbol{k}_o)\boldsymbol{K}_c(\boldsymbol{A}_o - \boldsymbol{b}_o\boldsymbol{k}_o)^T + \boldsymbol{b}_o\boldsymbol{b}_o^T. \tag{14}$$

If the coordinate transformation, $\tilde{\boldsymbol{x}}'(k) = \boldsymbol{T}^{-1}\tilde{\boldsymbol{x}}(k)$, is applied to the state-estimate feedback controller in (3), we obtain a new realization $(\tilde{\boldsymbol{D}}_o, \tilde{\boldsymbol{b}}_o, \tilde{\boldsymbol{g}}_o, \tilde{\boldsymbol{k}}_o)_n$ characterized by

$$\begin{aligned} \tilde{\boldsymbol{D}}_o &= \boldsymbol{T}^{-1}\boldsymbol{D}_o\boldsymbol{T}, & \tilde{\boldsymbol{b}}_o &= \boldsymbol{T}^{-1}\boldsymbol{b}_o \\ \tilde{\boldsymbol{g}}_o &= \boldsymbol{T}^{-1}\boldsymbol{g}_o, & \tilde{\boldsymbol{k}}_o &= \boldsymbol{k}_o\boldsymbol{T}. \end{aligned} \tag{15}$$

Notice that the transfer function from $[u(k), y(k)]^T$ to $r(k) - u(k) = \boldsymbol{k}_o\tilde{\boldsymbol{x}}(k)$ in (3) is invariant under such a coordinate transformation. The l_2-sensitivity measure in (10) is then changed to

$$\begin{aligned} m_2(\boldsymbol{T}) &= \text{tr}[\boldsymbol{T}^{-1}\boldsymbol{M}_1(\boldsymbol{T})\boldsymbol{T}^{-T}] + \Delta m_2(\boldsymbol{T}) \\ &= \text{tr}[\boldsymbol{T}^T\boldsymbol{S}_9(\boldsymbol{T})\boldsymbol{T}] + \Delta m_2(\boldsymbol{T}) \end{aligned} \tag{16}$$

and

$$\Delta m_2(\boldsymbol{T}) = \text{tr}[\boldsymbol{T}^T\boldsymbol{W}_9\boldsymbol{T}]+\text{tr}[\boldsymbol{T}^{-1}\boldsymbol{K}_1\boldsymbol{T}^{-T}]+\text{tr}[\boldsymbol{T}^T\boldsymbol{N}_9\boldsymbol{T}]$$

where $\boldsymbol{M}_1(\boldsymbol{T})$ and $\boldsymbol{S}_9(\boldsymbol{T})$ are obtained from the general expression below (10) as

$\boldsymbol{X}(z) = \boldsymbol{F}(z)W_1(z)\boldsymbol{G}(z)\boldsymbol{T}$ for $\boldsymbol{Y} = \boldsymbol{M}_1(\boldsymbol{T})$

$\boldsymbol{X}(z) = [\boldsymbol{T}^{-T}\boldsymbol{F}(z)W_1(z)\boldsymbol{G}(z)]^*$ for $\boldsymbol{Y} = \boldsymbol{S}_9(\boldsymbol{T})$.

For the new realization, (13) is changed to

$$\tilde{\boldsymbol{K}}_c = \boldsymbol{T}^{-1}\boldsymbol{K}_c\boldsymbol{T}^{-T}. \tag{17}$$

If l_2-scaling constraints are imposed on the new realization $(\tilde{\boldsymbol{D}}_o, \tilde{\boldsymbol{b}}_o, \tilde{\boldsymbol{g}}_o, \tilde{\boldsymbol{k}}_o)_n$ to suppress overflow, it is required that

$$(\boldsymbol{T}^{-1}\boldsymbol{K}_c\boldsymbol{T}^{-T})_{ii} = 1 \ \text{ for } \ i = 1, 2, \cdots, n. \tag{18}$$

As a result, we can formulate the problem of minimizing l_2-sensitivity under the l_2-scaling constraints as follows: *Given $\boldsymbol{A}_o$, $\boldsymbol{b}_o$, $\boldsymbol{c}_o$, $\boldsymbol{g}_o$, and $\boldsymbol{k}_o$, obtain an $n \times n$ nonsingular matrix $\boldsymbol{T}$ which minimizes the sensitivity measure $m_2(\boldsymbol{T})$ in (16) subject to the l_2-scaling constraints in (18).*

3. l_2-Sensitivity Minimization

The l_2-scaling constraints in (18) can be written as

$$(\hat{\boldsymbol{T}}^{-T}\hat{\boldsymbol{T}}^{-1})_{ii} = (\boldsymbol{T}^{-1}\boldsymbol{K}_c^{\frac{1}{2}}\boldsymbol{K}_c^{\frac{1}{2}}\boldsymbol{T}^{-T})_{ii} = 1 \quad \text{for } \ i = 1, 2, \cdots, n \tag{19}$$

where $\hat{\boldsymbol{T}} = \boldsymbol{T}^T\boldsymbol{K}_c^{-\frac{1}{2}}$. The conditions in (19) are always satisfied by choosing $\hat{\boldsymbol{T}}^{-1}$ as

$$\hat{\boldsymbol{T}}^{-1} = \left[\frac{\boldsymbol{t}_1}{||\boldsymbol{t}_1||}, \frac{\boldsymbol{t}_2}{||\boldsymbol{t}_2||}, \cdots, \frac{\boldsymbol{t}_n}{||\boldsymbol{t}_n||}\right]. \tag{20}$$

Applying matrix $\hat{\boldsymbol{T}}$ defined in (19) to (16), we can express the l_2-sensitivity measure in (16) as

$$\begin{aligned} J_o(\boldsymbol{x}) &= \text{tr}[\hat{\boldsymbol{T}}^{-T}\hat{\boldsymbol{M}}_1(\hat{\boldsymbol{T}})\hat{\boldsymbol{T}}^{-1}] + \Delta J_o(\boldsymbol{x}) \\ &= \text{tr}[\hat{\boldsymbol{T}}\hat{\boldsymbol{S}}_9(\hat{\boldsymbol{T}})\hat{\boldsymbol{T}}^T] + \Delta J_o(\boldsymbol{x}) \end{aligned} \tag{21}$$

with

$$\Delta J_o(\boldsymbol{x}) = \text{tr}[\hat{\boldsymbol{T}}\hat{\boldsymbol{W}}_9\hat{\boldsymbol{T}}^T]+\text{tr}[\hat{\boldsymbol{T}}^{-T}\hat{\boldsymbol{K}}_1\hat{\boldsymbol{T}}^{-1}]+\text{tr}[\hat{\boldsymbol{T}}\hat{\boldsymbol{N}}_9\hat{\boldsymbol{T}}^T]$$

where

$$\begin{aligned} &\boldsymbol{x} = (\boldsymbol{t}_1^T, \boldsymbol{t}_2^T, \cdots, \boldsymbol{t}_n^T)^T, \quad \hat{\boldsymbol{W}}_9 = \boldsymbol{K}_c^{\frac{1}{2}}\boldsymbol{W}_9\boldsymbol{K}_c^{\frac{1}{2}} \\ &\hat{\boldsymbol{M}}_1(\hat{\boldsymbol{T}}) = \boldsymbol{K}_c^{-\frac{1}{2}}\boldsymbol{M}_1(\boldsymbol{K}_c^{\frac{1}{2}}\hat{\boldsymbol{T}}^T\hat{\boldsymbol{T}}\boldsymbol{K}_c^{\frac{1}{2}})\boldsymbol{K}_c^{-\frac{1}{2}} \\ &\hat{\boldsymbol{S}}_9(\hat{\boldsymbol{T}}) = \boldsymbol{K}_c^{\frac{1}{2}}\boldsymbol{S}_9(\boldsymbol{K}_c^{\frac{1}{2}}\hat{\boldsymbol{T}}^T\hat{\boldsymbol{T}}\boldsymbol{K}_c^{\frac{1}{2}})\boldsymbol{K}_c^{\frac{1}{2}} \\ &\hat{\boldsymbol{K}}_1 = \boldsymbol{K}_c^{-\frac{1}{2}}\boldsymbol{K}_1\boldsymbol{K}_c^{-\frac{1}{2}}, \qquad \hat{\boldsymbol{N}}_9 = \boldsymbol{K}_c^{\frac{1}{2}}\boldsymbol{N}_9\boldsymbol{K}_c^{\frac{1}{2}}. \end{aligned}$$

Consequently, the problem of obtaining an $n \times n$ nonsingular matrix $\boldsymbol{T}$ which minimizes $m_2(\boldsymbol{T})$ in (16) subject to the l_2-scaling constraints in (18) can be converted into an unconstrained optimization problem of obtaining an $n^2 \times 1$ vector $\boldsymbol{x}$ which minimizes $J_o(\boldsymbol{x})$ in (21).

A quasi-Newton algorithm can be applied to minimize $J_o(\boldsymbol{x})$ in (21). Then in the kth iteration, the most recent point $\boldsymbol{x}_k$ is updated to point $\boldsymbol{x}_{k+1}$ as [12]

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \alpha_k\boldsymbol{d}_k \tag{22}$$

where

$$\begin{aligned} &\boldsymbol{d}_k = -\boldsymbol{S}_k\nabla J_o(\boldsymbol{x}_k), \quad \alpha_k = arg\ \min_{\alpha}\ J_o(\boldsymbol{x}_k + \alpha\boldsymbol{d}_k) \\ &\boldsymbol{S}_{k+1} = \boldsymbol{S}_k + \left(1+\frac{\boldsymbol{\gamma}_k^T\boldsymbol{S}_k\boldsymbol{\gamma}_k}{\boldsymbol{\gamma}_k^T\boldsymbol{\delta}_k}\right)\frac{\boldsymbol{\delta}_k\boldsymbol{\delta}_k^T}{\boldsymbol{\gamma}_k^T\boldsymbol{\delta}_k} - \frac{\boldsymbol{\delta}_k\boldsymbol{\gamma}_k^T\boldsymbol{S}_k+\boldsymbol{S}_k\boldsymbol{\gamma}_k\boldsymbol{\delta}_k^T}{\boldsymbol{\gamma}_k^T\boldsymbol{\delta}_k} \\ &\boldsymbol{S}_0=\boldsymbol{I}_n, \ \boldsymbol{\delta}_k=\boldsymbol{x}_{k+1}-\boldsymbol{x}_k, \ \boldsymbol{\gamma}_k=\nabla J_o(\boldsymbol{x}_{k+1})-\nabla J_o(\boldsymbol{x}_k) \end{aligned}$$

$\nabla J_o(\boldsymbol{x})$ is the gradient of $J_o(\boldsymbol{x})$ with respect to $\boldsymbol{x}$, and $\boldsymbol{S}_k$ is a positive-definite approximation of the inverse Hessian matrix of $J_o(\boldsymbol{x})$. In order to compute the value of α that minimizes $J_o(\boldsymbol{x}_k + \alpha\boldsymbol{d}_k)$, the Fletcher inexact line search method is used [13]. We choose a trivial initial point $\boldsymbol{x}_0$ obtained from an initial assignment $\hat{\boldsymbol{T}} = \boldsymbol{I}_n$ as a starting point, and continue the iteration process until

$$|J_o(\boldsymbol{x}_{k+1}) - J_o(\boldsymbol{x}_k)| < \varepsilon \tag{23}$$

is satisfied where $\varepsilon > 0$ is a prescribed tolerance.

In (22), the gradient of $J_o(\boldsymbol{x})$ can be efficiently evaluated using closed-form expressions as shown below.

$$\begin{aligned} &\boldsymbol{t}_j=(t_{1j}, t_{2j}, \cdots, t_{nj})^T \ \text{ for } \ j = 1, 2, \cdots, n \\ &\nabla J_o(\boldsymbol{x})=\left[\frac{\partial J_o(\hat{\boldsymbol{T}})}{\partial t_{11}}, \cdots, \frac{\partial J_o(\hat{\boldsymbol{T}})}{\partial t_{n1}}, \cdots, \frac{\partial J_o(\hat{\boldsymbol{T}})}{\partial t_{nn}}\right]^T \end{aligned} \tag{24}$$

with

$$\begin{aligned} \frac{\partial J_o(\hat{\boldsymbol{T}})}{\partial t_{ij}} &= \lim_{\Delta\to 0}\frac{J_o(\hat{\boldsymbol{T}}_{ij}) - J_o(\hat{\boldsymbol{T}})}{\Delta} \\ &= 2(\beta_1 - \beta_2 + \beta_3 - \beta_4 + \beta_5) \end{aligned} \tag{25}$$

where $\hat{\boldsymbol{T}}_{ij}$ is the matrix obtained from $\hat{\boldsymbol{T}}$ with a perturbed (i, j)th component and is given by

$$\begin{aligned} &\hat{\boldsymbol{T}}_{ij} = \hat{\boldsymbol{T}} + \frac{\Delta\hat{\boldsymbol{T}}\boldsymbol{g}_{ij}\boldsymbol{e}_j^T\hat{\boldsymbol{T}}}{1 - \Delta\boldsymbol{e}_j^T\hat{\boldsymbol{T}}\boldsymbol{g}_{ij}}, \qquad \hat{\boldsymbol{T}}_{ij}^{-1} = \hat{\boldsymbol{T}}^{-1} - \Delta\boldsymbol{g}_{ij}\boldsymbol{e}_j^T \\ &\boldsymbol{g}_{ij} = \partial\left\{\frac{\boldsymbol{t}_j}{||\boldsymbol{t}_j||}\right\}/\partial t_{ij} = \frac{1}{||\boldsymbol{t}_j||^3}(t_{ij}\boldsymbol{t}_j - ||\boldsymbol{t}_j||^2\boldsymbol{e}_i) \\ &\beta_1 = \boldsymbol{e}_j^T\hat{\boldsymbol{T}}\hat{\boldsymbol{M}}_1(\hat{\boldsymbol{T}})\hat{\boldsymbol{T}}^T\hat{\boldsymbol{T}}\boldsymbol{g}_{ij} \\ &\beta_2 = \boldsymbol{e}_j^T\hat{\boldsymbol{T}}^{-T}\hat{\boldsymbol{S}}_9(\hat{\boldsymbol{T}})\boldsymbol{g}_{ij}, \qquad \beta_3 = \boldsymbol{e}_j^T\hat{\boldsymbol{T}}\hat{\boldsymbol{W}}_9\hat{\boldsymbol{T}}^T\hat{\boldsymbol{T}}\boldsymbol{g}_{ij} \\ &\beta_4 = \boldsymbol{e}_j^T\hat{\boldsymbol{T}}^{-T}\hat{\boldsymbol{K}}_1\boldsymbol{g}_{ij}, \qquad \beta_5 = \boldsymbol{e}_j^T\hat{\boldsymbol{T}}\hat{\boldsymbol{N}}_9\hat{\boldsymbol{T}}^T\hat{\boldsymbol{T}}\boldsymbol{g}_{ij}. \end{aligned}$$

The algorithm was applied to quite a number of simulation examples and fast convergence was observed in all the cases.

4. Numerical Example

As a numerical example, a linear discrete-time system is specified by [4]

$$\boldsymbol{A}_o = \begin{bmatrix} 0 & 0 & 0.775585 \\ 1 & 0 & -2.534177 \\ 0 & 1 & 2.758200 \end{bmatrix}^T, \quad \boldsymbol{b}_o = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$\boldsymbol{c}_o = \begin{bmatrix} 0.0022 & 0.0044 & 0.0022 \end{bmatrix}.$$

When the state observer has poles at $z = 0.4532, 0.5761$ and 0.8437 and when the poles of the regulator are placed at $z = 0.9067, 0.7523$ and 0.6231, we obtain

$$\boldsymbol{k}_o = \begin{bmatrix} 0.350562 & -0.818344 & 0.476100 \end{bmatrix}$$
$$\boldsymbol{g}_o = 10^2 \begin{bmatrix} 0.818859 & 1.010891 & 1.182995 \end{bmatrix}^T.$$

We applied a coordinate transformation matrix given by $\boldsymbol{T}_s = \mathrm{diag}\{21.2378, 21.2378, 21.2378\}$ so as to satisfy the l_2-scaling constraints such that $(\boldsymbol{T}_s^{-1}\boldsymbol{K}_c\boldsymbol{T}_s^{-T})_{ii} = 1$ for $i = 1, 2, \cdots, n$. Then the l_2-sensitivity measure in (10) was computed as

$$m_2 = 9.649719 \times 10^4.$$

By choosing $\hat{\boldsymbol{T}} = \boldsymbol{I}_3$ as an initial assignment in (22) and $\varepsilon = 10^{-8}$ as a tolerance in (23), the quasi-Newton algorithm took 20 iterations to converge to the solution

$$\hat{\boldsymbol{T}}^{opt} = \begin{bmatrix} 1.172374 & -2.210847 & -5.362575 \\ -1.523614 & -2.345467 & -6.184722 \\ -2.559976 & -1.101154 & -0.516404 \end{bmatrix}$$
$$\boldsymbol{T}^{opt} = \begin{bmatrix} -3.099470 & -5.362382 & -2.573048 \\ -3.696822 & -5.776983 & -2.403745 \\ -4.234706 & -6.172099 & -2.218043 \end{bmatrix}.$$

The minimized l_2-sensitivity measure in (21) was found to be

$$J_o(\boldsymbol{x}) = 7.8701104396.$$

5. Conclusion

The l_2-sensitivity of a closed-loop transfer function to the coefficients of a state-estimate feedback controller has been analyzed, and the problem of minimizing the l_2-sensitivity measure subject to l_2-scaling constraints has been formulated. An iterative algorithm based on a quasi-Newton method has been developed for synthesizing the optimal structure of a state-estimate feedback controller with minimum l_2-sensitivity and no overflow. Computer simulation results have demonstrated the validity and effectiveness of the proposed technique.

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