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A Study on Frequency Entrainment in Externally Excited Phase Oscillator – Operation of excitation based on energy –

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Abstract—This paper studies frequency entrainment in externally excited phase oscillator from the viewpoint of energy. The energy aspects originate from the fact that oscillators exchange the stored energy through coupling to fall in synchronism. Averaged equation is derived by using the Jacobi elliptic functions. Numerical and analytical results are shown with respect to phase difference and energy exchange. The obtained analytical result allows us to extract the central operation from the excitation. In this paper, it appears that frequency entrainment occurs through operation of excitation to sustain the rotational motion and regulate velocity in the phase oscillator.

1. Introduction

Frequency entrainment has been extensively investigated in nonlinear dynamical systems [1, 2, 3]. The phenomenon appears when coupled oscillators fall in synchronism with each other. Energy aspects have been applied to the analyses of frequency entrainment [4, 5, 6, 7, 8]. The viewpoint originates from the fact that oscillators exchange the stored energy through coupling to synchronize. The energy exchange is regarded as an essential feature to understand synchronization including frequency entrainment.

In this paper, we analyze frequency entrainment in an externally excited phase oscillator based on energy exchange. The dynamics of the system is described by the following equation:

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \omega, \tag{1a}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\gamma\omega - \sin\phi + N + A\sin\Omega t, \tag{1b}$$

where γ denotes the damping coefficient, N the constant torque, A and Ω the amplitude and frequency of excitation, respectively. The phase oscillator is analogous to many familiar nonlinear dynamical systems such as mechanical pendulum, Josephson junction circuit [9], phase-locked loop [10], and power system [11]. Hence frequency entrainment also appears in these systems and the entrained state is represented by the stable rotation.

The Jacobi elliptic functions are known as appropriate functions which can depict rotation. Indeed some works have been reported which analyze rotational motion of forced pendulum [12, 13]. The authors utilized the Jacobi elliptic functions on the averaging of the rotational motion. The functions make it possible to formulate an approximated solution for the rotation.

This paper studies frequency entrainment in a phase oscillator excited by harmonic torque from a viewpoint of energy. Averaged equation is derived through the Jacobi elliptic functions. Numerical and analytical results are shown with respect to phase difference and energy exchange. On the basis of these results, we discuss how the excitation induces the rotational motion representing frequency entrainment.

2. Averaging formed by elliptic functions

This section describes an averaging for rotational motion of the phase oscillator. The approximated solution is formulated based on the Jacobi elliptic functions.

2.1. Approximated solution

In this section, we first formulate an approximated solution for rotational motion of the phase oscillator by using Jacobi elliptic functions. An energy aspect determines the formula of solution. The stored energy of phase oscillator, H, is defined at a state (ϕ, ω) as

$$H(\phi, \omega) \triangleq \frac{1}{2}\omega^2 - \cos\phi. \tag{2}$$

When $H > 1$, the phase oscillator rotates. Rotational motion which conserves the stored energy H is focused on here. The motion is represented by a closed orbit, called constant-energy surface, in the cylindrical phase space [1]. By introducing a dependent variable ξ with the stored energy H fixed, that is, $\mathrm{d}H/\mathrm{d}\xi = 0$, we can describe the dynamics as follows:

$$\Omega_0\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = \frac{\partial}{\partial\omega}H(\phi, \omega) = \omega, \tag{3a}$$

$$\Omega_0\frac{\mathrm{d}\omega}{\mathrm{d}\xi} = -\frac{\partial}{\partial\phi}H(\phi, \omega) = -\sin\phi. \tag{3b}$$

Where Ω_0 denotes the self-rotatory frequency of phase oscillator. The stored energy H can vary independently of the variable ξ. Hence we assume another dependent variable a which is defined as

$$a \triangleq \sqrt{\frac{H+1}{2}} = \sqrt{\frac{1}{4}\omega^2 + \sin^2\frac{\phi}{2}}. \qquad (4)$$

When $a > 1$, rotational motions appear.

Substituting Eq. (4) into Eq. (3a) gives a definition of the Jacobi elliptic function $\mathrm{sn}\,(a\xi/\Omega_0, 1/a)$ [14]:

$$\frac{a}{\Omega_0}\xi = \int_0^{\phi} \frac{\mathrm{d}(\phi/2)}{\sqrt{1-a^{-2}\sin^2(\phi/2)}} = \mathrm{sn}^{-1}\left(\sin\frac{\phi}{2}, \frac{1}{a}\right). \qquad (5)$$

An inverse function of the integral in Eq. (5) is defined by

$$\phi = 2\,\mathrm{am}\left(\frac{a}{\Omega_0}\xi, \frac{1}{a}\right), \qquad (6)$$

where $\mathrm{am}\,(a\xi/\Omega_0, 1/a)$ is the Jacobi amplitude [14]. Eqs. (3a) and (6) gives a representation of ω:

$$\omega = \Omega_0\frac{\partial\phi}{\partial\xi} = 2a\,\mathrm{dn}\left(\frac{a}{\Omega_0}\xi, \frac{1}{a}\right), \qquad (7)$$

where $\mathrm{dn}\,(a\xi/\Omega_0, 1/a)$ is also the Jacobi elliptic function [14]. Here we introduce a constant a_0 as a value of a for the periodic rotation of unexcited phase oscillator. The self-rotatory frequency Ω_0 is a function of a_0. Considering $\xi = \Omega t + \theta$ and the difference between a and a_0 determines an approximated solution:

$$\phi = 2\,\mathrm{am}\left(\frac{a_0}{\Omega_0}(\Omega t+\theta), \frac{1}{a_0}\right), \qquad (8a)$$

$$\omega = 2a\,\mathrm{dn}\left(\frac{a_0}{\Omega_0}(\Omega t+\theta), \frac{1}{a_0}\right). \qquad (8b)$$

The variable θ corresponds to the phase difference between the rotational motion and the excitation.

2.2. Averaged equation

Averaged equation is derived for rotational motion of the phase oscillator in this section. By substituting the approximated solution (8) into Eq. (1) and averaging it, we obtain the following averaged equation:

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \Omega_0 - \Omega - \Theta_\beta(\theta, a, \alpha), \qquad (9a)$$

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{\pi N - 4\gamma a E_0}{4K_0} - \frac{\pi A}{4K_0\cosh(\pi K_0'/K_0)}\sin\theta, \qquad (9b)$$

where $K_0 = K(1/a_0)$ and $E_0 = E(1/a_0)$ are elliptic integrals of the first and second kind, respectively. In addition, $K_0' = K'(1/a_0) = K([1-1/a_0^2]^{1/2})$. The obtained averaged equation (9) depends on the constant α which originates from the integral range on averaging. In Eq. (9a) the term Θ_β implies the influence of excitation upon the time derivative of θ, which we call the external action, and is decomposed into

$$\Theta_0(\theta, a) = \frac{\pi A}{4aK_0\cosh(\pi K_0'/K_0)}\cos\theta, \qquad (10)$$

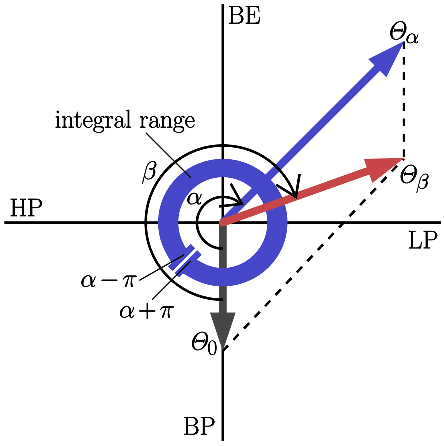

Figure 1: Schematic diagram of external action on phase dynamics determined by the integral range. Each arrow depicts the phasor of external action. Θ_0 denotes the fixed component of external action, Θ_α the component dependent on the integral range $[\alpha-\pi, \alpha+\pi]$, and Θ_β the phasor sum of the above component. α and β represent the direction of Θ_α and Θ_β. The ring reveals the integral range on the averaging. BP, HP, LP, and BE symbols frequency response characteristics for the direction of Θ_β: band-pass, low-pass, high-pass, and band-elimination.

and

$$\Theta_\alpha(\theta, a, \alpha) = \frac{\pi A}{4aK_0}\cos(\theta-\alpha). \qquad (11)$$

Fig. 1 shows the relationship between α and the external action Θ_β or the components Θ_0, Θ_α.

3. Determination of Integral Range

This section gives an appropriate value to the integral range on averaging. Numerical response curves guide the decision of analytical response curves for the valid integral range. We here define numerically calculated phase θ and external energy $\mathcal{W}$. The variable θ corresponds to the phase difference, which implies

$$\theta = \langle \phi - \Omega t \rangle, \qquad (12)$$

where $\langle\cdot\rangle$ denotes time average. External energy $\mathcal{W}$ is defined as

$$\mathcal{W} = -\frac{\pi A a\sin\theta}{K_0\cosh(\pi K_0'/K_0)} \qquad (13)$$

for the analytical study and

$$\mathcal{W} = \langle \omega A \sin\Omega t \rangle \qquad (14)$$

for the numerical study. The external energy $\mathcal{W}$ implies the work done on the oscillator by the excitation physically. In the following discussion, γ is set at 0.5 and N at 1.

Numerical response curves are studied to determine the appropriate integral range through observation. Fig. 2

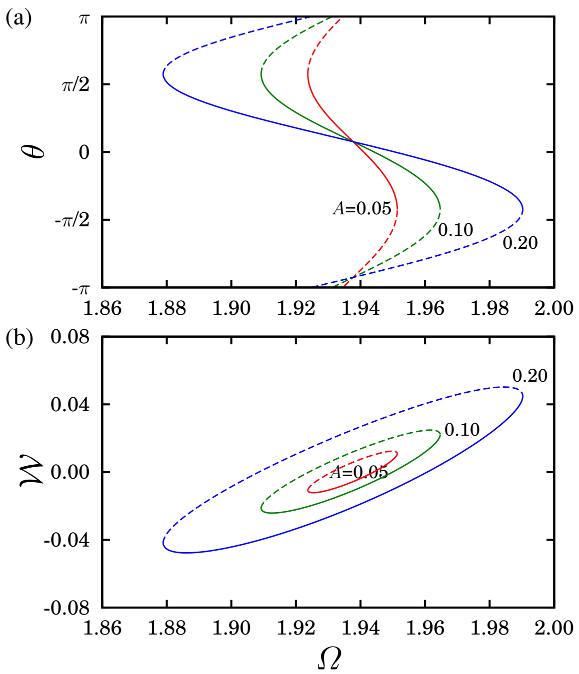

Figure 2: Response curves of phase θ and external energy $\mathcal{W}$ obtained numerically at $\gamma = 0.5$ and $N = 1$. The solid and dashed lines depict stable and unstable periodic solutions, respectively.

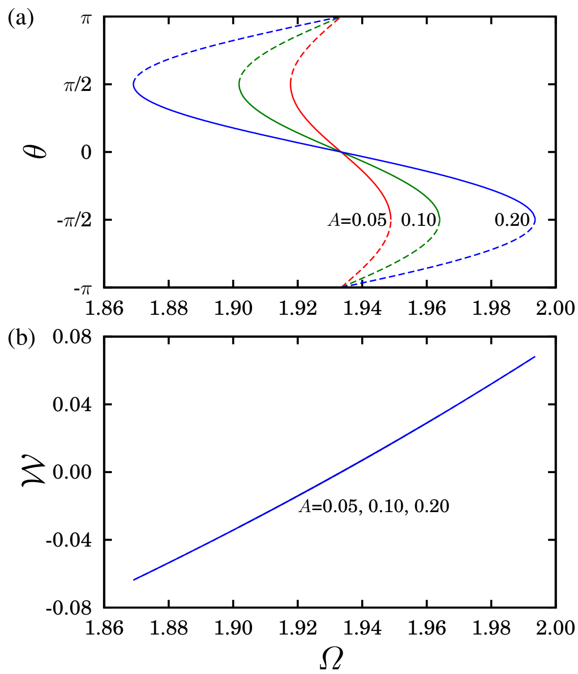

Figure 3: Response curves of phase θ and external energy $\mathcal{W}$ obtained analytically at $\gamma = 0.5$ and $N = 1$ with $\alpha = \pi - \mathrm{asin}[\tanh(\pi K_0'/K_0)]$. The solid and dashed lines illustrate stable and unstable equilibria, respectively.

shows the phase θ and the external energy $\mathcal{W}$ calculated numerically. The solid and dashed lines illustrate stable and unstable solutions, respectively. The response curves of the phase θ have two boundaries related to the stability at $\theta = \pm\pi/2 + \epsilon$, where ϵ is a small positive. The response curves of external energy $\mathcal{W}$ show the characteristics: monotonically increasing with the excitation frequency Ω and antiresonance.

By considering the above numerical result, we adopt the condition at $\alpha = \pi - \mathrm{asin}[\tanh(\pi K_0'/K_0)]$. Fig. 3 shows response curves obtained analytically at the condition. The range of phase θ completely characterizes the stability. Stable solution exists in the range $(-\pi/2, \pi/2)$ and unstable solution in the rest. The analytical response curves of phase θ is consistent with the numerical one. On the other hand, for the external energy $\mathcal{W}$ the analytical result captures the characteristic of monotonic increase with the excitation frequency Ω.

4. Discussion

In this section, we discuss the operation of excitation which induces frequency entrainment: sustaining the rotational motion and regulating the velocity. The obtained analytical result allows us to extract the central operation from the excitation.

An essential roles of the excitation is to sustain the entrained rotational motion. We focus on the case at $\theta = 0$ which implies the situation $\Omega = \Omega_0$ to eliminate the other operation of excitation, that is, regulation of rotational velocity. At the case $\theta = 0$, the excitation operates on the phase oscillator in the rotational direction when $\phi \in [0, \phi)$ shown by Fig. 4(a), and in the inverse direction when $\phi \in [-\pi, 0)$ shown by Fig. 4(b). This behavior of excitation attenuates the restoring force $-\sin\phi$. Because the restoring force inhibits the rotational motion of phase oscillator, this behavior implies that the excitation assists the rotational motion. On the other hand, the excitation cannot sustain the rotational motion in $|\theta| \gg 0$. Thus, in θ is close to 0 the excitation sustains the rotational motion of phase oscillator. In addition, the operation does not produce any external energy $\mathcal{W}$ at $\theta = 0$ shown by Fig. 3.

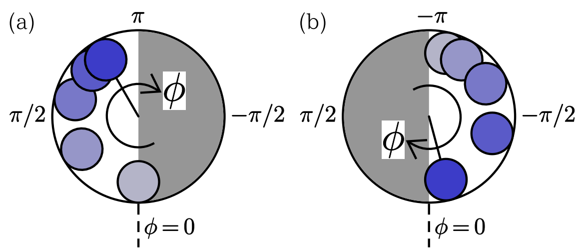

Figure 4: Two intervals in rotational motion of the phase oscillator; (a) $\phi \in [0, \pi)$, (b) $\phi \in [-\pi, 0)$.

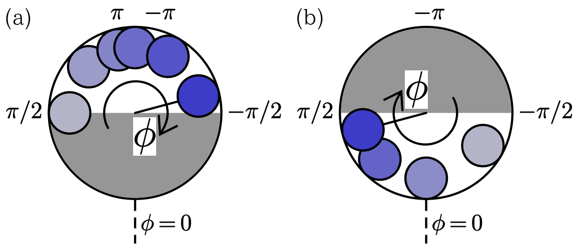

Figure 5: Two intervals in rotational motion; (a) $\phi \in [\pi/2, \pi)$ and $\phi \in [-\pi, -\pi/2)$, (b) $\phi \in [-\pi/2, \pi/2)$.

The other substantial operation of the excitation is the regulation of the rotational velocity corresponding to frequency entrainment. This point can be explained in an energy aspect. In the situation of $\Omega \neq \Omega_0$ the phase oscillator requires energy supply or energy loss induced by the excitation because the phase oscillator keeps energy balance at only $\Omega = \Omega_0$. Fig. 3 shows that the external energy $\mathcal{W}$ is negative in $\theta \in (0, \pi)$ and positive in $\theta \in (-\pi, 0)$. Hence the energy balance is kept through the shift of θ. We give a physical interpretation to the relationship between the external energy $\mathcal{W}$ and the phase θ. At $\theta = \pi/2$, the excitation operates on the phase oscillator in the rotational direction when $\phi \in [\pi/2, \pi)$ and $\phi \in [-\pi, -\pi/2)$ shown by Fig. 5(a), and in the inverse direction when $\phi \in [-\pi/2, \pi/2)$ shown by Fig. 5(b). That is, the excitation supplies energy to the oscillator in Fig. 5(a), and withdraws energy from the oscillator in Fig. 5(b). Energy loss in Fig. 5(a) is smaller than energy supply in Fig. 5(b) because the rotational velocity in Fig. 5(a) is smaller. The same discussion is applied to the case at $\theta = -\pi/2$. For another θ the total energy produced by the excitation is determined by the magnitude of the shift of θ. Thus the operation of excitation is appropriate to the physical interpretations at $\alpha = \pi - \mathrm{asin}[\tanh(\pi K'_0/K_0)]$.

5. Concluding Remarks

This paper discussed frequency entrainment in the phase oscillator from an aspect of energy. Averaged equation for rotation was derived through the Jacobi elliptic functions while the equation possesses an arbitrary property originated from integral range of averaging. In this paper, the appropriate integral range is determined by observation on numerical and analytical results. Then we discussed the entrained rotational motion based on energy produced by the excitation. The excitation entrains the phase oscillator with the rotational motion sustained. When the excitation frequency is different from the self-rotatory frequency, the excitation regulates the rotational velocity and keeps energy balance with the phase shift. Thus, in the phase oscillator, frequency entrainment appears through the operation of excitation to sustain the rotational motion and regulate the velocity.

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