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Symmetric Oscillations in Three LC Ladder Circuit

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Abstract—This paper describes an approach to finding out patterns of oscillations using symmetries. In order to clarify oscillations in a three-phase circuit systematically, we derive higher symmetric circuit, which is a three LC ladder circuit. Using the symmetry of the circuit, we classify periodic and almost periodic oscillations and construct the lattice of those oscillations. The lattice enables to predict typical oscillations in the circuit.

1. Introduction

The symmetric three phase circuit shown in Fig.1 is a fundamental model of power systems. The nonlinearity of the delta-connected inductors generates many kinds of nonlinear oscillations, e.g., subharmonic oscillations[1], asymmetric oscillations[4], cnoidal waves[2] and ILMs[3]. Although the detail of bifurcations of each oscillation is analyzed by the homotopy method[4], relations of each nonlinear oscillation are not clear.

In order to find out the oscillations in the three-phase circuit systematically, we consider higher symmetric circuit, which is a three LC ladder circuit. The higher symmetries of the circuit enables to list the patterns of oscillations and predict typical oscillations.

First, we show the symmetry of the three LC ladder circuit and classify the periodic oscillations with respect to the symmetries[5]. Next, we extend the method to almost periodic oscillations. Further, we confirm that the higher symmetric oscillations are generated in the circuit from the observation of the global phase space.

2. Three-Phase Circuit And Three LC Ladder Circuit

The equation of the three-phase circuit shown in Fig.1 is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \boldsymbol{\psi}_{\mathrm{abc}} \\ \boldsymbol{u}_{\mathrm{abc}} \end{pmatrix} = \begin{pmatrix} -\boldsymbol{A}_{\mathrm{abc}}\boldsymbol{u}_{\mathrm{abc}} \quad +\boldsymbol{e}_{\mathrm{abc}} - \boldsymbol{R}_{\mathrm{abc}}\boldsymbol{i}_{\mathrm{abc}} \\ \boldsymbol{A}_{\mathrm{abc}}^{\mathrm{T}}\boldsymbol{i}_{\mathrm{abc}} \end{pmatrix}. \tag{1}$$

$$\boldsymbol{A}_{\mathrm{abc}} \triangleq \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix},$$

$$\begin{aligned} \boldsymbol{\psi}_{\mathrm{abc}} &\triangleq (\psi_{\mathrm{a}}, \psi_{\mathrm{b}}, \psi_{\mathrm{c}})^{\mathrm{T}} \\ \boldsymbol{u}_{\mathrm{abc}} &\triangleq (u_{\mathrm{a}}, u_{\mathrm{b}}, u_{\mathrm{c}})^{\mathrm{T}} \\ \boldsymbol{R}_{\mathrm{abc}} &\triangleq \boldsymbol{A}_{\mathrm{abc}}^{\mathrm{T}}\boldsymbol{A}_{\mathrm{abc}}R + \boldsymbol{I}r \\ \boldsymbol{i}_{\mathrm{abc}}(\boldsymbol{\psi}_{\mathrm{abc}}) &\triangleq (i(\psi_{\mathrm{a}}), i(\psi_{\mathrm{b}}), i(\psi_{\mathrm{c}}))^{\mathrm{T}} \\ \boldsymbol{e}_{\mathrm{abc}}(t) &\triangleq E_{\mathrm{abc}}(\sin(\omega_e t), \sin(\omega_e t - \tfrac{2\pi}{3}), \sin(\omega_e t + \tfrac{2\pi}{3}))^{\mathrm{T}}, \end{aligned}$$

where $\boldsymbol{\psi}_{\mathrm{abc}}$ and $\boldsymbol{u}_{\mathrm{abc}}$ are the flux interlinkages of the inductors and voltages of the capacitors, respectively. The $\boldsymbol{I}$ denotes unit matrix and $(*)^{\mathrm{T}}$ denotes transposition. R, r, ω_e represents normalized circuit parameters which corresponds to Y-connected resistors, Δ-connected resistors, and angular frequency of the voltage sources, respectively. We assume that the characteristics of the flux interlinkages $i(\psi)$ is represented by monotone increasing odd function.

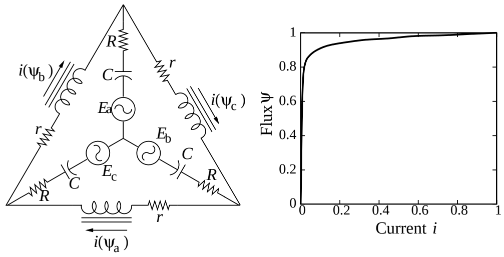

Figure 1: Three-phase circuit

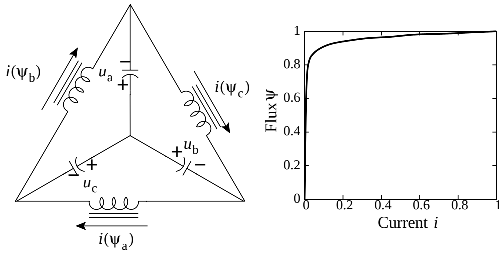

Figure 2: Three LC ladder circuit

In order to find out oscillations in the three-phase circuit, we derive higher symmetric circuit, which is a three LC ladder circuit shown in Fig. 2, by removing the resistors and voltage sources from the three-phase circuit. The equation of the circuit is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \boldsymbol{\psi}_{\mathrm{abc}} \\ \boldsymbol{u}_{\mathrm{abc}} \end{pmatrix} = \begin{pmatrix} -\boldsymbol{A}_{\mathrm{abc}}\boldsymbol{u}_{\mathrm{abc}} \\ \boldsymbol{A}_{\mathrm{abc}}^{\mathrm{T}}\boldsymbol{i}_{\mathrm{abc}} \end{pmatrix}. \tag{2}$$

For simplicity, we rewrite Eq.(2) by

$$\frac{\mathrm{d}\boldsymbol{x}_{\mathrm{abc}}}{\mathrm{d}t} = \boldsymbol{f}_{\mathrm{abc}}(\boldsymbol{x}_{\mathrm{abc}}), \qquad \boldsymbol{x}_{\mathrm{abc}} = (\boldsymbol{\psi}_{\mathrm{abc}}^{\mathrm{T}},\ \boldsymbol{u}_{\mathrm{abc}}^{\mathrm{T}})^{\mathrm{T}}. \tag{3}$$

3. Symmetries of Three LC Ladder Circuit

In order to describe the symmetry of the three LC ladder circuit, we introduce the following permutations $\check{\gamma}$:

$$\check{\gamma} = \begin{pmatrix} \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \chi_{\mathrm{a}} & \chi_{\mathrm{b}} & \chi_{\mathrm{c}} \end{pmatrix}, \quad \begin{array}{l} \chi_n \in \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}\ (n \in \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}), \\ \chi_n \neq \chi_m\ (n \neq m \in \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}). \end{array} \quad .$$

For example, cyclic permutation is represented by

$$\check{c}_3 \triangleq \begin{pmatrix} \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \mathrm{b} & \mathrm{c} & \mathrm{a} \end{pmatrix}. \tag{4}$$

The action $\check{c}_3$ satisfies the following commutativity:

$$\check{c}_3 \boldsymbol{f}_{\mathrm{abc}}(\boldsymbol{x}_{\mathrm{abc}}) = \boldsymbol{f}_{\mathrm{abc}}(\check{c}_3 \boldsymbol{x}_{\mathrm{abc}}). \tag{5}$$

This relation shows that the three LC ladder circuit has cyclic symmetry. Next, we consider the reflection.

$$\check{\sigma}_{\mathrm{a}} \triangleq \begin{pmatrix} \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \mathrm{a} & \mathrm{c} & \mathrm{b} \end{pmatrix}. \tag{6}$$

The action $\check{\sigma}_{\mathrm{a}}$ satisfies the following commutativity:

$$\check{\sigma}_{\mathrm{a}} \boldsymbol{f}_{\mathrm{abc}}(\boldsymbol{x}_{\mathrm{abc}}) = \boldsymbol{f}_{\mathrm{abc}}(\check{\sigma}_{\mathrm{a}} \boldsymbol{x}_{\mathrm{abc}}). \tag{7}$$

This relation shows that the three LC ladder circuit has reflection symmetry. It is noted that the three-phase circuit does not have the symmetry due to the three-phase source. Further, we consider inversion symmetry based on the odd symmetry of the function $i(\psi)$:

$$\tilde{\boldsymbol{x}}_{\mathrm{abc}} = \check{i} \boldsymbol{x}_{\mathrm{abc}}. \tag{8}$$
$$\tilde{\boldsymbol{\psi}}_{\mathrm{abc}} = -\boldsymbol{\psi}_{\mathrm{abc}}, \ \tilde{\boldsymbol{u}}_{\mathrm{abc}} = -\boldsymbol{u}_{\mathrm{abc}} \ .$$

The action $\check{i}$ satisfies

$$\check{i} \boldsymbol{f}_{\mathrm{abc}}(\boldsymbol{x}_{\mathrm{abc}}) = \boldsymbol{f}_{\mathrm{abc}}(\check{i} \boldsymbol{x}_{\mathrm{abc}}). \tag{9}$$

This relation shows that the three LC ladder circuit has inversion symmetry.

From the 3 symmetries, the three LC ladder circuit has the symmetry with respect to the group $\check{\Gamma}$

$$\check{\Gamma} \triangleq \left\{ \begin{array}{cccccc} \check{e}, & \check{c}_3, & \check{c}_3^2, & \check{i}, & \check{i}\check{c}_3, & \check{i}\check{c}_3^2, \\ \check{\sigma}_{\mathrm{a}}, & \check{\sigma}_{\mathrm{b}}, & \check{\sigma}_{\mathrm{c}}, & \check{i}\check{\sigma}_{\mathrm{a}}, & \check{i}\check{\sigma}_{\mathrm{b}}, & \check{i}\check{\sigma}_{\mathrm{c}} \end{array} \right\}. \tag{10}$$

Subgroups of the group $\check{\Gamma}$ is listed in Tab.1 and the lattice of the subgroups are shown in Fig.3.

4. Classification of Periodic Oscillations

4.1. Spatio-temporal symmetry and spatial symmetry

We consider period T oscillations in the three LC ladder circuit which satisfies

$$\boldsymbol{x}_{\mathrm{abc}}(t) = \boldsymbol{x}_{\mathrm{abc}}(t + T). \tag{11}$$

Table 1: Subgroups of Group $\check{\Gamma}$

order 1	$\check{E}$	$\triangleq \{\check{e}\}$
order 2	$\check{I}$	$\triangleq \{\check{e},\ \check{i}\}$
	$\check{V}$	$\triangleq \{\check{e},\ \check{\sigma}_{\mathrm{a}}\}$
	$\check{J}$	$\triangleq \{\check{e},\ \check{\sigma}_{\mathrm{a}}\check{i}\}$
order 4	$\check{\Gamma}^{(4)}$	$\triangleq \{\check{e},\ \check{i},\ \check{\sigma}_{\mathrm{a}},\ \check{\sigma}_{\mathrm{a}}\check{i}\}$
order 3	$\check{C}_3$	$\triangleq \{\check{e},\ \check{c}_3,\ \check{c}_3^2\}$
order 6	$\check{C}_3 \times \check{I}$	$\triangleq \{\check{e},\ \check{i},\ \check{c}_3,\ \check{c}_3\check{i},\ \check{c}_3^2,\ \check{c}_3^2\check{i}\}$
	$\check{C}_3 \times \check{V}$	$\triangleq \{\check{e},\ \check{c}_3,\ \check{c}_3^2,\ \check{\sigma}_{\mathrm{a}},\ \check{\sigma}_{\mathrm{b}},\ \check{\sigma}_{\mathrm{c}}\}$
	$\check{C}_3 \times \check{J}$	$\triangleq \{\check{e},\ \check{c}_3,\ \check{c}_3^2,\ \check{\sigma}_{\mathrm{a}}\check{i},\ \check{\sigma}_{\mathrm{b}}\check{i},\ \check{\sigma}_{\mathrm{c}}\check{i}\}$
order 12	$\check{\Gamma}$	$\triangleq \left\{ \begin{array}{c} \check{e},\ \check{c}_3,\ \check{c}_3^2,\ \check{i},\ \check{i}\check{c}_3,\ \check{i}\check{c}_3^2, \\ \check{\sigma}_{\mathrm{a}},\ \check{\sigma}_{\mathrm{b}},\ \check{\sigma}_{\mathrm{c}},\ \check{i}\check{\sigma}_{\mathrm{a}},\ \check{i}\check{\sigma}_{\mathrm{b}},\ \check{i}\check{\sigma}_{\mathrm{c}} \end{array} \right\}$

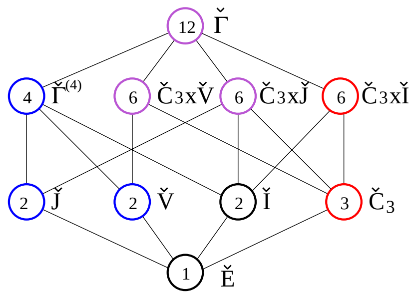

Figure 3: Lattice of subgroups of $\check{\Gamma}$

We normalize the time by the period T and consider period 2π oscillation $\hat{\boldsymbol{x}}(\theta) : \mathbb{T}^1 \mapsto \mathbb{R}^6$, where $\mathbb{T}^1$ denotes 1-torus.

Let us consider a subgroup $\check{H} \subset \check{\Gamma}$. If a periodic oscillation $\hat{\boldsymbol{x}}(\theta)$ satisfies

$$\check{H} = \left\{ \check{\gamma} \in \check{\Gamma} \mid \check{\gamma}\{\hat{\boldsymbol{x}}(\theta)\} = \{\hat{\boldsymbol{x}}(\theta)\} \right\} \tag{12}$$

for all the actions $\check{\gamma} \in \check{H}$, the periodic oscillation has spatio-temporal symmetry [5]. This relation shows that the $\check{H}$-action preserves the trajectory of $\hat{\boldsymbol{x}}(\theta)$ and an action $\check{\gamma} \in \check{H}$ causes only a shift k:

$$^{\forall}\theta \in \mathbb{T}^1, \ \check{\gamma}\hat{\boldsymbol{x}}(\theta) = \hat{\boldsymbol{x}}(\theta - k). \tag{13}$$

We represent the correspondence between $\check{\gamma}$ and k by a map $k = \Theta(\check{\gamma})$. Then, the kernel of the map Θ is defined by

$$\check{K} \triangleq \left\{ \check{\gamma} \in \check{H} \mid \Theta(\check{\gamma}) = 0 \right\}. \tag{14}$$

The kernel $\check{K}$ derives a fixed-point subspace

$$\mathrm{Fix}(\check{K}) \triangleq \{\boldsymbol{x} \in \mathbb{R}^6 \mid \check{\gamma}\boldsymbol{x} = \boldsymbol{x}, \ ^{\forall}\check{\gamma} \in \check{K}\} \subset \mathbb{R}^6. \tag{15}$$

In this sence, the subgroup $\check{K}$ represents spatial symmetry of the three LC ladder circuit. In order to exist periodic oscillations, it is necessary that the $\check{H}/\check{K}$ is isomorphic to cyclic group $\check{C}_m$ and that the dimension of $\mathrm{Fix}(\check{K})$ is not less than 2 [5].

Table 2: Classification of periodic oscillations

$\check{H}$	$\check{K}$	label
$\check{\Gamma}$	$\check{\Gamma}$	$\boldsymbol{o}$
$\check{C}_3 \times \check{I}$	$\check{E}$	$\mathrm{osc}^{\mathrm{e}}(\mathrm{c,i})$
$\check{C}_3$	$\check{C}_3$	$\mathrm{osc}^{\mathrm{c}}(\mathrm{c})$
$\check{C}_3$	$\check{E}$	$\mathrm{osc}^{\mathrm{e}}(\mathrm{c})$
$\check{\Gamma}^{(4)}$	$\check{V}$	$\mathrm{osc}^{\mathrm{v}}(4)$
$\check{\Gamma}^{(4)}$	$\check{J}$	$\mathrm{osc}^{\mathrm{j}}(4)$
$\check{V}$	$\check{V}$	$\mathrm{osc}^{\mathrm{v}}(\mathrm{v})$
$\check{V}$	$\check{E}$	$\mathrm{osc}^{\mathrm{e}}(\mathrm{v})$
$\check{J}$	$\check{J}$	$\mathrm{osc}^{\mathrm{j}}(\mathrm{j})$
$\check{J}$	$\check{E}$	$\mathrm{osc}^{\mathrm{e}}(\mathrm{j})$
$\check{I}$	$\check{E}$	$\mathrm{osc}^{\mathrm{e}}(\mathrm{i})$
$\check{E}$	$\check{E}$	$\mathrm{osc}^{\mathrm{e}}(\mathrm{e})$

Table 3: Pattern of oscillations

label	phase($\boldsymbol{\psi}_{\mathrm{abc}}$)			comments
$\mathrm{osc}^{\mathrm{e}}(\mathrm{c,i})$ M_3	$\xi(\theta)$	$\xi(\theta-k)$	$\xi(\theta-2k)$	$\xi(\theta) = -\xi(\theta+\pi), k \in \{+\mathrm{k}_3, -\mathrm{k}_3\}$
$\mathrm{osc}^{\mathrm{c}}(\mathrm{c})$	$\xi(\theta)$	$\xi(\theta)$	$\xi(\theta)$	
$\mathrm{osc}^{\mathrm{e}}(\mathrm{c})$	$\xi(\theta)$	$\xi(\theta-k)$	$\xi(\theta-2k)$	$k \in \{+\mathrm{k}_3, -\mathrm{k}_3\}$
$\mathrm{osc}^{\mathrm{v}}(4)$ M_1	$\eta(\theta)$	$\xi(\theta)$	$\xi(\theta)$	$\xi(\theta) = -\xi(\theta+\pi), \eta(\theta) = -\eta(\theta+\pi)$
$\mathrm{osc}^{\mathrm{j}}(4)$ M_2	0	$\xi(\theta)$	$-\xi(\theta)$	$\xi(\theta) = -\xi(\theta+\pi)$
$\mathrm{osc}^{\mathrm{v}}(\mathrm{v})$	$\eta(\theta)$	$\xi(\theta)$	$\xi(\theta)$	
$\mathrm{osc}^{\mathrm{e}}(\mathrm{v})$	$\eta(\theta)$	$\xi(\theta)$	$\xi(\theta+\pi)$	$\eta(\theta) = \eta(\theta+\pi)$
$\mathrm{osc}^{\mathrm{j}}(\mathrm{j})$	0	$\xi(\theta)$	$-\xi(\theta)$	
$\mathrm{osc}^{\mathrm{e}}(\mathrm{j})$	$\eta(\theta)$	$\xi(\theta)$	$-\xi(\theta+\pi)$	$\eta(\theta) = -\eta(\theta+\pi)$
$\mathrm{osc}^{\mathrm{e}}(\mathrm{i})$	$\boldsymbol{x}_{\mathrm{abc}}(\theta) = -\boldsymbol{x}_{\mathrm{abc}}(\theta+\pi)$			

Based on the above conditions, we can classify symmetric periodic oscillations with respect to the subgroups $\check{H}$ and $\check{K}$. The classified oscillations are listed in Tab.2 and the patterns of the oscillations are shown in Tab.3, where $\eta, \xi : \mathbb{T}^1 \mapsto \mathbb{R}^1$ are period 2π functions, $\mathrm{k}_3 \equiv \frac{2}{3}\pi$, and M_1, M_2 and M_3 denotes typical highly symmetric oscillations. Further, the lattice of symmetric periodic oscillation on 3 LC ladder circuit with respect to $\check{H}$ is shown in Fig.4.

4.2. Higher symmetric oscillations

Let us consider the typical higher symmetric oscillations M_1, M_2 and M_3 shown in Figs. 5, 6 and 7, respectively. M_1 and M_2 are symmetric with respect to the reflection and the inversion. The difference between M_1 and M_2 comes from the spatial symmetry. The M_1 corresponds to single-phase oscillations in the three-phase circuit and the M_2 is unstable. In the sence of ILM, the M_1 and M_2 correspond to ST mode and Page mode, respectively[6]. The M_3 is symmetric with respect to the cyclic and inversion symmetry. The $k = +\mathrm{k}_3$ and $k = -\mathrm{k}_3$ fix the propagating directions.

The symmetrical coordinates transform and the $0\alpha\beta$ coordinates transform for the symmetric three-phase circuit correspond to the cyclic and the reflection symmetries, respectively. The M_3 with $+k_3$ and $-k_3$ corresponds to positive-phase-sequence and negative-phase-sequence component, respectively. The M_1 and M_2 corresponds to α component and β component, respectively.

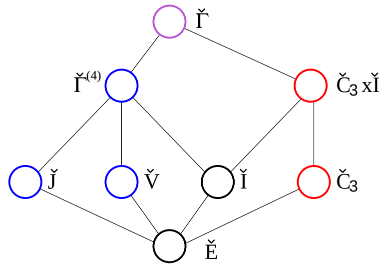

Figure 4: Lattice of symmetric periodic oscillations.

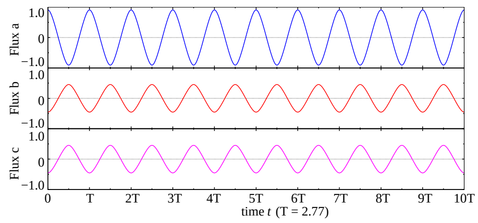

Figure 5: M_1 has symmetry w.r.t. reflection.

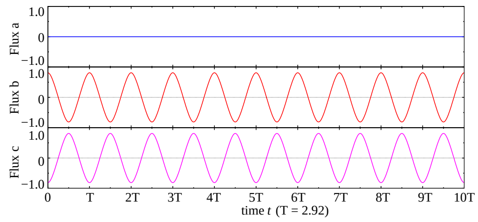

Figure 6: M_2 has symmetry w.r.t. reflection.

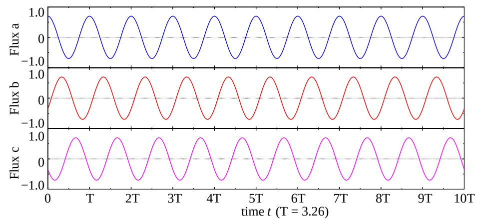

Figure 7: M_3 has symmetry w.r.t. cyclic symmetry.

5. Almost periodic oscillation

5.1. Definition

We extend the method of the classification of the periodic oscillations to almost periodic oscillations. We define the almost periodic oscillation with normalized phase $\boldsymbol{\theta}$ by $\hat{\boldsymbol{x}}(\boldsymbol{\theta}) : \mathbb{T}^2 \mapsto \mathbb{R}^6$. Then, a subgroup $\check{H}$ is defined by

$$\check{H} = \left\{ \check{\gamma} \in \check{\Gamma} \mid \check{\gamma}\{\hat{\boldsymbol{x}}(\boldsymbol{\theta})\} = \{\hat{\boldsymbol{x}}(\boldsymbol{\theta})\} \right\} \tag{16}$$

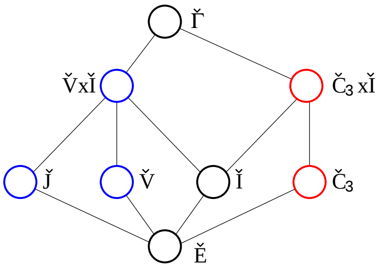

Figure 8: Lattice of almost periodic oscillations.

for all the action $\check{\gamma} \in \check{H}$. The $\check{H}$-action preserves the trajectory of $\hat{\boldsymbol{x}}(\boldsymbol{\theta})$ and an action $\check{\gamma}$ causes only a shift $\boldsymbol{k} \in \mathbb{T}^2$:

$$^{\forall}\boldsymbol{\theta} \in \mathbb{T}^2,\ \check{\gamma}\hat{\boldsymbol{x}}(\boldsymbol{\theta}) = \hat{\boldsymbol{x}}(\boldsymbol{\theta} - \boldsymbol{k}). \tag{17}$$

This relation defines a map $\boldsymbol{\Theta}(\check{\gamma}) : \check{H} \mapsto \mathbb{T}^2$ and the kernel of the map $\boldsymbol{\Theta}(\check{\gamma})$ is defined by

$$\check{K} \triangleq \left\{\check{\gamma} \in \check{H} \mid \boldsymbol{\Theta}(\check{\gamma}) = \boldsymbol{o}\right\}. \tag{18}$$

The subgroup $\check{K}$ defines the fixed-point subspace Eq.(15). The condition that $\boldsymbol{\Theta}$ is a group homomorphism is described by

$$\check{H}/\check{K} \simeq \check{C}_{m1} \times \check{C}_{m2}, \tag{19}$$

where $m1 \in \mathbb{Z}$ is a divisor of $m2$. Additionally, Fix($\check{K}$) is not less than 4. Based on the conditions, we can illustrate the lattice of symmetric almost periodic oscillations shown in Fig.8. The higher symmetric waveforms beat(v,i) and beat(c,i) which belongs to $\check{V} \times \check{I}$ and $\check{C}_3 \times \check{I}$ respectively are shown in Figs. 9 and 10.

In order to confirm oscillations in the three LC ladder circuit, we calculate Poincare map of the cross section

$$\Sigma \equiv \left\{(\psi_\alpha, u_\alpha, \psi_\beta, u_\beta) | u_\alpha = 0, \psi_\alpha > 0\right\}, \tag{20}$$

where the suffixes α and β represents α and β coordinate in $0\alpha\beta$ coordinates. Although the original phase space is 6-dimension, assuming that 0-phase component is equal to 0 and fixing the Hamiltonian $H = 0.83$, all the phase space is projected into 2-dimensional plane. From the Poincare map shown in Fig.11, we can confirm that periodic oscillations M_1, M_2 and M_3 exist and the almost periodic oscillation beat(v,i) and beat (c,i) exist around the M_1 and M_3, respectively. Almost all the phase space is covered by the 3 regions of beat(v,i) and 2 regions of beat(c,i).

5.2. Conclusion

We classify the periodic and almost periodic oscillations in the three LC ladder circuit and derive the lattice of those oscillations. As typical oscillations in the circuit, the higher symmetric oscillations are confirmed by the observation of the global phase space.

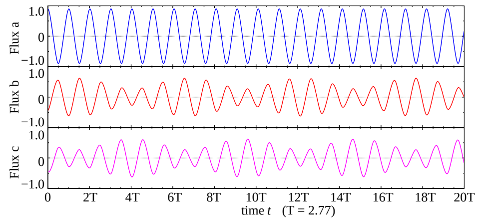

Figure 9: Beat(v,I) has symmetry w.r.t. reflection.

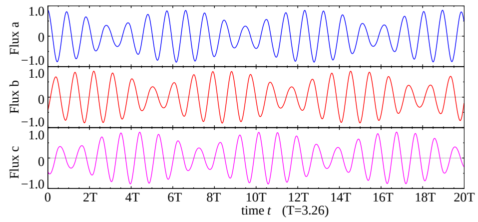

Figure 10: Beat(c,i) has symmetry w.r.t. cyclic symmetry.

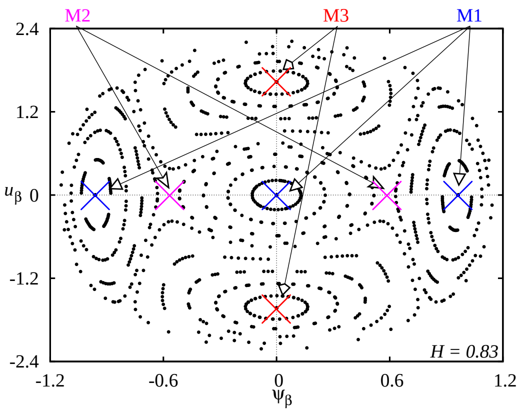

Figure 11: Poincare map

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