2009 International Symposium on Nonlinear Theory and its Applications
NOLTA'09, Sapporo, Japan, October 18-21, 2009

Chaos control of chaotic Hindmarsh–Rose models for neurons

Ioan Grosu[†], Servilia Oancea[‡]

† Bioengineering, University of Medicine and Pharmacy Gr. T. Popa, Iasi, Romania
‡ Biophysics, University of Agricultural Sciences and Veterinary Medicine, Iasi , Romania
Email: ioan.grosu@instbi.umfiasi.ro, liaoancea@yahoo.com

Abstract – Master-slave and mutual method for synchronization and amplification (attenuation) of chaos for two chaotic systems are presented. Numerical results are given for the synchronization of two Hindmarsh-Rose models for neurons.

1. Introduction

One of the possible applications of the synchronization is in the neuroscience because chaotic regimens have been observed experimentally in neurons. The time evolution of the neuronal activity is commonly describe by nonlinear diferential equations. The Hindmarsh-Rose model, one of the most popular neuron models exhibiting complicated dynamics, is a three-variable model for the bursting of neurons and a variant of the FitzHugh-Nagumo model [1]. Hindmarsh and Rose work was initiated by the discovery of a neuronal cell in the brain of pond snail *Lymnae* which, when it was depolarized by a short pulse, generated an action potential followed by slow depolarizing after-potential. In invertebrates there are networks of neurons in which every neuron has reciprocal connections to other members. The Hindmarsh-Rose equations are developed to study synchronization of firing of two snail neurons without the need to use the full Hodgkin-Huxley equations. The natural choice was to use the FitzHugh-Nagumo model, which is more or less a simplification of the Hodgkin-Huxley equations. FitzHugh and Nagumo observed independently that in the Hodgkin-Huxley equations, the membrane potential as well as sodium activation evolves on similar time-scales during an action potential, while sodium inactivation and potassium activation change on similar, although slower time scales. Starting with the two-variable subsystem which describes the action potential, they add a third variable which is governed by a simple linear equation, and note that the resulting autonomous system admits aperiodic behavior. Then the complicated current-voltage relationship of the conductance models has been replaced by polynomials in the dynamical variables.
The dynamical behavior of the Hindmarsh–Rose model has been intensively investigated in the last two decades. In recent years, more and more authors have devoted to investigating the synchronization of the Hindmarsh–Rose models from numerical simulation studies [2]. Recently Wu et al. [3] investigated the issues of control and synchronization of the chaotic Hindmarsh–Rose models via impulsive control with varying impulsive intervals and Ma at al. [4] the role of noise in the biological and neuronal system because the neurons often are sensitive to the external noise. Jackson and Grosu [5], [6] developed a powerful method of control: the open-plus-closed-loop (OPCL) method. This method gives precise driving for any continuous system in order to reach any desired dynamics and it has been applied to synchronization of two identical systems by Lerescu et al. [7], and Oancea [8]. More than this it can be extends to 3 systems and 4 systems [9], [10].
The main objective of this paper is to investigate the synchronization and amplification (attenuation) of chaos for Hindmarsh–Rose models of neurons.

2. Master-slave Synchronization

Let's consider a general master system:

$$d\mathbf{X}/dt = \mathbf{F}(\mathbf{X});\ \mathbf{X} \in \mathbf{R}^n \qquad (1)$$

then the slave system:

$$d\mathbf{x}/dt = \mathbf{F}(\mathbf{x}) + \mathbf{D}(\mathbf{x},\mathbf{X}) \qquad (2)$$

where $\mathbf{D}(\mathbf{x},\mathbf{X}) = (\mathbf{A} - \partial \mathbf{F}/\partial \mathbf{x}|_{\mathbf{x}=\mathbf{X}})(\mathbf{x} - \mathbf{X}) - 1/2(\partial^2 \mathbf{F}/\partial \mathbf{x}^2)(\mathbf{x} - \mathbf{X})^2 - 1/6(\partial^3 \mathbf{F}/\partial \mathbf{x}^3)(\mathbf{x} - \mathbf{X})^3 + \ldots$ assures $\mathbf{x}(t) \rightarrow \mathbf{X}(t)$ for any $\|\mathbf{x}(0) - \mathbf{X}(0)\|$ small enough.

A is a constant Hurwitz matrix with negative real part eigenvalues. The matrix A should be chosen in such a manner in order that the coupling to be as simple as possible.

3. Mutual Synchronization

Let's consider two identical general oscillators:

$$d\mathbf{x}/dt = \mathbf{F}(\mathbf{x});\ d\mathbf{y}/dt = \mathbf{F}(\mathbf{y}); \qquad (3)$$

The coupled systems are:

$$d\mathbf{x}/dt = \mathbf{F}(\mathbf{x}) + \mathbf{u}(\mathbf{x},\mathbf{y});\quad d\mathbf{y}/dt = \mathbf{F}(\mathbf{y}) + \mathbf{u}(\mathbf{x},\mathbf{y}); \qquad (4)$$

where $\mathbf{u}(\mathbf{x},\mathbf{y}) = (A - d\mathbf{F}(\mathbf{s})/d\mathbf{s}) * (\mathbf{x} - \mathbf{y})/2$, $\mathbf{s} = (\mathbf{x} + \mathbf{y})/2$ and A is the Hurwitz matrix.

The present method has been applied to all systems from the Sprott collection [11].

4. Amplification of chaos

Grosu et al. [12] designed the coupling for stable synchronization and antisynchronization in chaotic systems under parameter mismatch.
The driver is:

$dy/dt = \mathbf{F(y)} + \Delta\mathbf{F(y)};$ (5)

$\mathbf{y} \in \mathbf{R}^n$, unde $\Delta\mathbf{F(y)}$ contains mismatch parameters.

The driven system is given by:

$dx/dt = \mathbf{F(x)} + \mathbf{D(x,\alpha y)}$ (6)

where $\mathbf{D(x,\alpha y)} = \alpha\, dy/dt - \mathbf{F}(\alpha \mathbf{y}) - (\mathbf{A} - J\, \mathbf{F}(\alpha \mathbf{y}))(\mathbf{x} - \alpha \mathbf{y})$ (7)

J being the Iacobian and A the arbitrary constant Hurwitz matrix.

5. Numerical results

The Hindmarsh-Rose model has three variables X_1, X_2, X_3, satisfying the following polynomial equations:

$dX_1/dt = X_2 - aX_1^3 + bX_1^2 - X_3 + I;$

$dX_2/dt = c - dX_1^2 - X_2;$ (8)

$dX_3/dt = r\,[s(X_1 - e) - X_3]$

Here X_1 is membrane action potential, X_2 is potential of the ionic channels subserving accommodation, X_3 the slow adaptation current which moves the voltage in and out of the inherent bistable regime and which terminates spike discharges, *I*, external direct current and a, b, c, d, e, r, s are constants.

Depending on the values of above parameters neurons can be in a steady state, they can generate a periodic low-frequency repetitive firing, chaotic bursts, or high frequency discharges of action potentials. Just as proposed in [1], we consider the system (8) with the constant values:

$a=1;\ b=3;\ c=1;\ d=5;\ I=3.25;\ e=-(1+\sqrt{5})/2;\ r=0.005;\ s=4$

The Hindmarsh-Rose system is chaotic (Fig. 1):

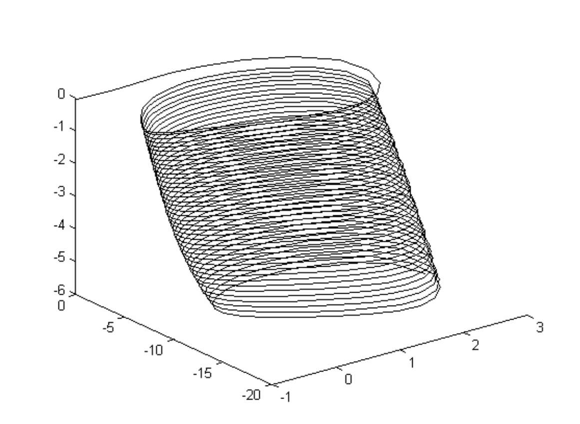

Fig.1. Phase portrait for Hindmarsh-Rose system with initial conditions ($X_1(0) = -1$; $X_2(0) = -0.1$ $X_3(0) = -0.01$)

We can choose the Hurvitz matrix having a two constant parameters and the Routh-Hurwitz conditions give for these parameters:

$0.02 > 0.05(p_1+p_2)$

$0.025 > 1.005p_1 + p_2$

The slave system (of the master system (8)) with $p_1=p_2=-10$ is:

$dx_1/dt = x_2 - x_1^3 + 3x_1^2 - x_3 + 3.25 + (-10 + 3X_1^2 - 6X_1)(x_1 - X_1);$

$dx_2/dt = 1 - 5x_1^2 - x_2 + (-10 + 10X_1))(x_1 - X_1);$

$dx_3/dt = 0.005[4(x_1 + 1.618) - x_3]$ (9)

Fig.2 and 3 presents the fast synchronization of two Hindmarsh-Rose systems for $p_1=p_2=-10$]

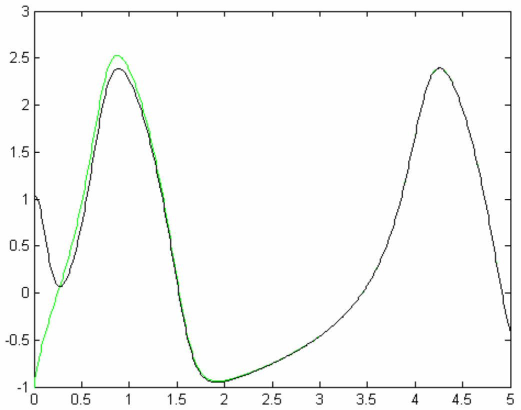

Fig.2 $X_1(t)$-green $x_1(t)$-black [$X_1(0) = -1$; $X_2(0) = -0.1$ $X_3(0) = -0.01$; $x_1(0) = 1$; $x_2(0) = 0.1$ $x_3(0) = 0.01$]

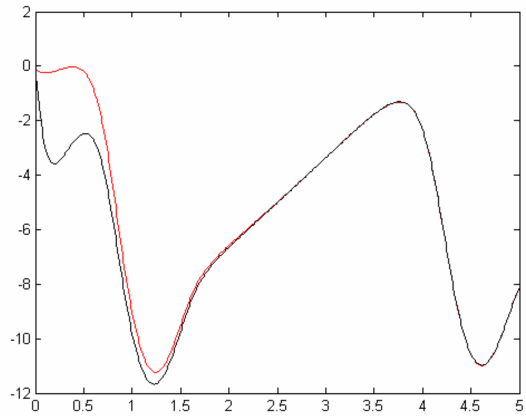

Fig.3. $X_2(t)$-red, $x_2(t)$-black [$X_1(0) = -1$; $X_2(0) = -0.1$ $X_3(0) = -0.01$; $x_1(0) = 1$; $x_2(0) = 0.1$ $x_3(0) = 0.01$; $p_1=p_2=-10$]

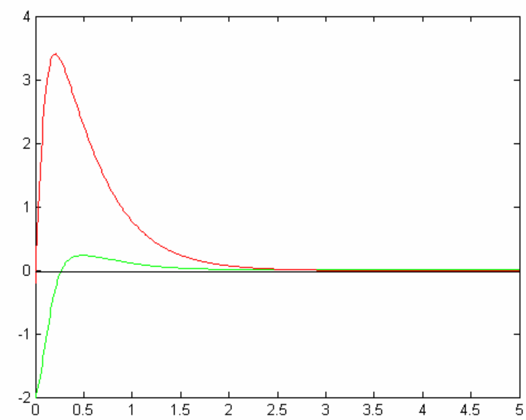

Fig.4. Synchronization errors between (8) and (9) [$X_1(t)$-$x_1(t)$, green ; $X_2(t)$- $x_2(t)$, red ; $X_3(t)$- $x_3(t)$, black] and $p_1=p_2=-10$; ($X_1(0)=-1$; $X_2(0)=-0.1$; $X_3(0)=-0.01$; $x_1=1$; $x_2=0.1$; $x_3=0.01$)

The mutual synchronization systems are:

$$dx_1/dt = x_2 - x_1^3 + 3x_1^2 - x_3 + 3.25 + [-10 + 3(x_1+y_1)^2/4 - 6(x_1+y_1)/2]\,(x_1-y_1)/2;$$
$$dx_2/dt = 1 - 5x_1^2 - x_2 + [-10 + 10(x_1+y_1)/2]\,(x_1-y_1)/2\ ;$$
$$dx_3/dt = 0.005\,[4(x_1+1.618) - x_3]$$
$$dy_1/dt = y_2 - y_1^3 + 3y_1^2 - y_3 + 3.25 + [-10 + 3(x_1+y_1)^2/4 - 6(x_1+y_1)/2]\,(-x_1+y_1)/2;$$
$$dy_2/dt = 1 - 5y_1^2 - y_2 + [-10 + 10(x_1+y_1)/2]\,(-x_1+y_1)/2\ ;$$
$$dy_3/dt = 0.005\,[4(y_1+1.618) - y_3]$$

Numerical results for the mutual synchronization are given in Fig.5, 6 and 7.

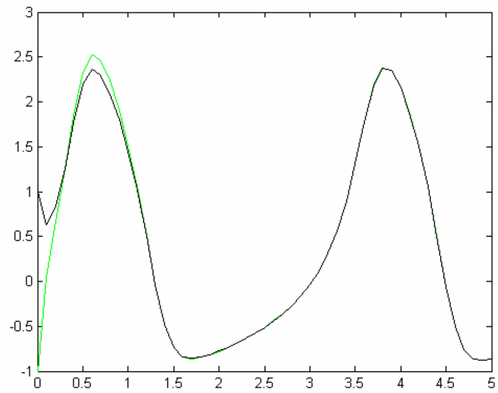

Fig.5 $x_1(t)$-green, $y_1(t)$-black [$x_1(0) = -1$; $x_2(0) = -0.1$ $x_3(0) = -0.01$; $y_1(0) = 1$; $y_2(0) = 0.1$ $y_3(0) = 0.01$; $p_1 = p_2 = -10$]

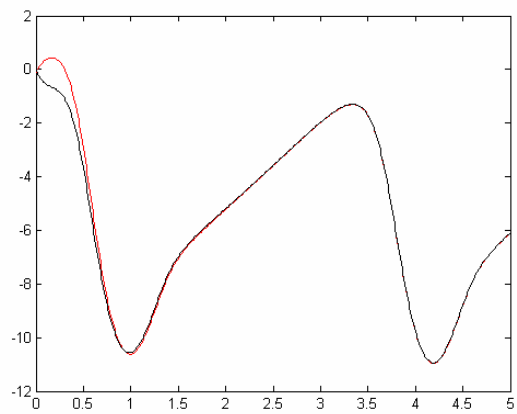

Fig.6 $x_2(t)$-red, $y_2(t)$-black [$x_1(0) = -1$; $x_2(0) = -0.1$ $x_3(0) = -0.01$; $y_1(0) = 1$; $y_2(0) = 0.1$ $y_3(0) = 0.01$; $p_1 = p_2 = -10$]

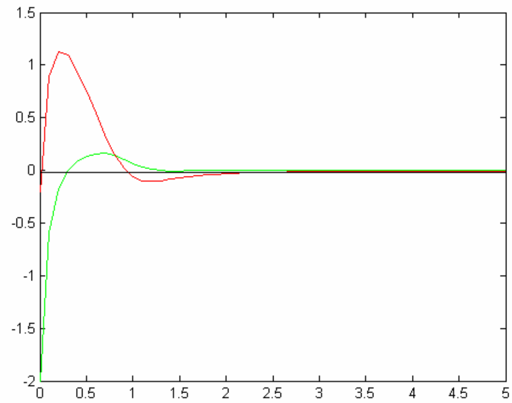

Fig. 7 Synchronization errors [($x_1(t) - y_1(t)$, green ; $x_2(t) - y_2(t)$, red ; $x_3(t)$ - $y_3(t)$, black)] oscillators [$x_1(0) = -1$;$x_2(0) = -0.1$; $x_3(0) = -0.01$; $y_1(0) = 1$; $y_2(0) = 0.1$; $y_3(0) = 0.01$; $p_1 = p_2 = -10$]

The driver system 5 and the driven system (6) with $\alpha = -2$ are:

$$dy_1/dt = y_2 - y_1^3 + 3y_1^2 - y_3 + 3.25 - 0.5\,y_1 + 0.1y_2 - 0.1y_3;$$
$$dy_2/dt = 1 - 5y_1^2 - y_2\ ;$$
$$dy_3/dt = 0.005\,[4(y_1+1.618) - y_3]$$
$$dx_1/dt = x_2 - x_1^3 + 3x_1^2 - x_3 + 3.25 + y_1^3 - 0.2\,y_2 + 0.2y_3 - 6\,y_1^3 - 18\,y_1^2 + [-10 - 12\,y_1^2 - 12y_1\,](x_1+2y_1)$$
$$dx_2/dt = 1 - 5x_1^2 - x_2 + 30\,y_1^2 + [-10 - 20y_1]\,(x_1+2y_1)\ ;$$
$$dx_3/dt = 0.005\,[4(x_1+1.618) - x_3]$$

Amplification of chaos for Hindmarsh-Rose system is given in figures 8-11.

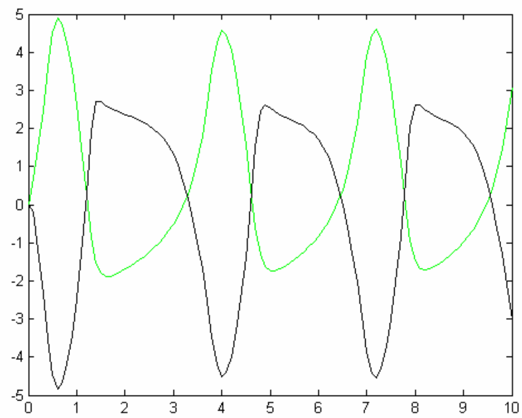

Fig. 8 $2y_1(t)$-green, $x_1(t)$-black, for $\alpha = -2$; ($y_1(0) = 0.001$; $y_2(0) = 0.001$; $y_3(0) = 0.001$; $x_1(0) = -0.001$; $x_2(0) = 0.001$; $x_3(0) = 0.001$)

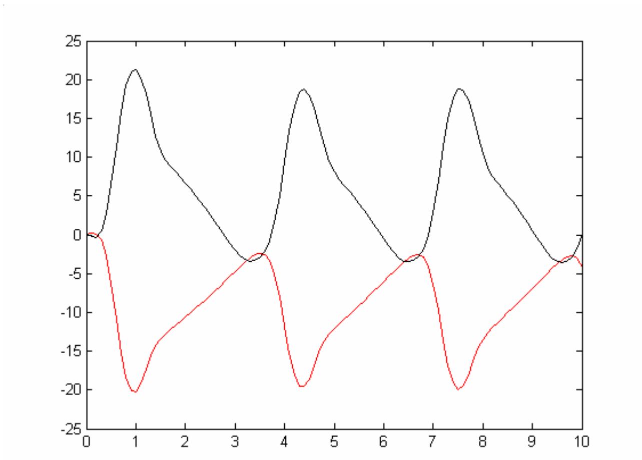

Fig. 9 $2y_2(t)$-red, $x_2(t)$-black, for α=-2; ($y_1(0)$=0.001; $y_2(0)$= 0.001; $y_3(0)$= 0.001; $x_1(0)$=- 0.001; $x_2(0)$= 0.001; $x_3(0)$= 0.001)

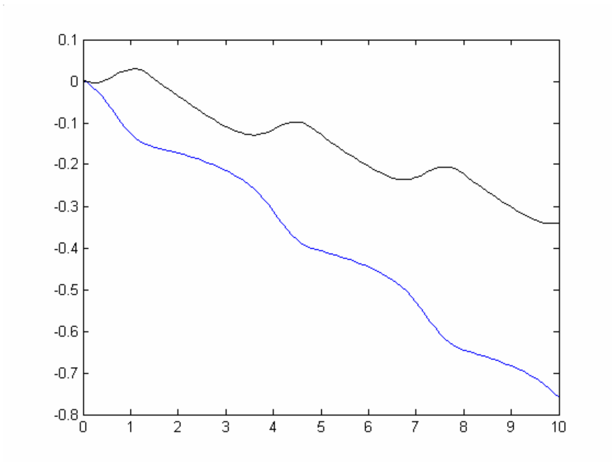

Fig. 10 $2y_3(t)$-blue $x_3(t)$-black, for α=-2; ($y_1(0)$=0.001; $y_2(0)$= 0.001; $y_3(0)$= 0.001; $x_1(0)$=- 0.001; $x_2(0)$= 0.001; $x_3(0)$= 0.001)

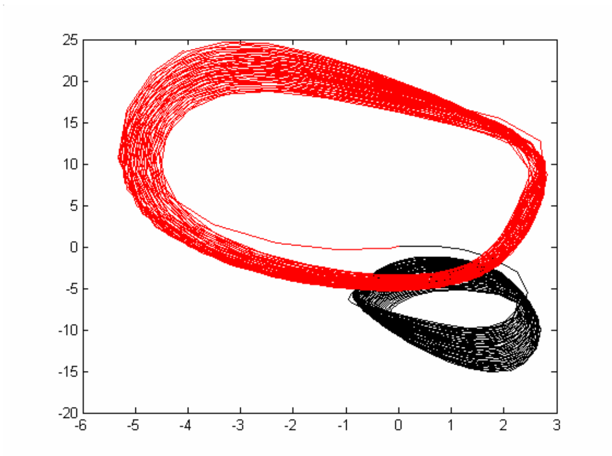

Fig. 11 Phase portrait of (y_1, y_2)-black and (x_1, x_2)-red, for p=-10 and α=-2; ($y_1(0)$=0.001; $y_2(0)$= 0.001; $y_3(0)$= 0.001; $x_1(0)$=- 0.001; $x_2(0)$= 0.001; $x_3(0)$= 0.001)

6. Conclusions

In this paper we applied the master-slave, mutual synchronization and amplification of chaos methods to the chaotic systems that are known as Hindmarsh-Rose oscillators. The transient time until synchronization depends on initial conditions of two systems and on the values of negative part of eigenvalues. Generally speaking, synchronization properties of dynamical neural networks essentially depend on the coupling configuration, the number of cells, and the type of coupling.

References

[1] J.C. Alexander, D. Cai, "On the Dynamics of Bursting Systems", 1990. http://www.case.edu/artsci/math/alexander/pdf/cai.pdf

[2] H.Yu, J.Peng, Chaotic synchronization and control in nonlinear-coupled Hindmarsh-Rose neural systems, *Chaos, Solitons and Fractals*, 29, pp. 342-348, 2006.

[3] Q.Wu, J.Zhou, L.Xiang, Z.Liu, "Impulsive control and synchronization of chaotic Hindmarsh-Rose models for neural activity", *Chaos, Solitons and Fractals*, 2008. http://www.sciencedirect.com/science?_ob=MImg&_imagekey=B6TJ4-4V1FBPM-1-7&_cdi=5300&_user=6260515&_orig=search&_coverDate=11%2F28%2F2008&_sk=999999999&view=c&wchp=dGLzVtz-zSkWb&md5=b39403ee8dc4014c9502c4477a9c8c7e&ie=/sdarticle.pdf,

[4] J. Ma, Y. Jia, J. Tang ,L-J. Yang, "Breakup of Spiral Waves in Coupled Hindmarsh–Rose Neurons", *Chin Phys Lett*, 25(12), pp.4325-4328, 2008.

[5] E. A. Jackson, I. Grosu, **"**An Open-Plus-Closed-Loop (OPCL) Control of Complex Dynamic Systems, *Physica* D, 85(1) pp.1-9, 1995.

[6] I. Grosu, "Robust Synchronization", *Phys. Rev*. 56, pp.3709-3712, 1997.

[7] A.I. Lerescu, N. Constandache, S. Oancea, I.Grosu, "Collection of master-slave synchronized chaotic systems", *Chaos, Solitons and Fractals*, 22(3), pp.599-604, 2004.

[8] S.Oancea, "Synchronization of chaotic electronic Sprott's circuits", *Journal of Optoelectronics and Advanced materials*, 7(6), pp.2919-2923, 2005.

[9] I.Grosu, "General Coupling for Synchronization of 3 Idendical Oscillators", *Int. J. of Bifurcation and Chaos*, 17 (10), pp.3519-3522, 2007.

[10] I.Grosu, A. Lerescu**,** S.Oancea, **"**Synchronization of 4 identical chaotic systems", 3rd International Conference Physics and Control (PhysCon 2007), 2007. http://lib.physcon.ru/?item=1288

[11] A.I. Lerescu, S. Oancea, I. Grosu, "Collection of Mutually Synchronized Chaotic Systems", *Physics Letters A*, 352, pp.222-228, 2006.

[12] I. Grosu, E. Padmanaban, P. K. Roy, S. K. Dana, "Designing Coupling for Synchronization and Amplification of Chaos", *Phys Rev Lett* 100, pp.234102, 1-4, 2008.