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Performance of Adiabatic Quantum Computation using Neuron-like Interconnections

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Abstract—Quantum computation algorithms indicate possibility that non-deterministic polynomial time problems can be solved much faster than classical methods. Farhi *et al.* have proposed an adiabatic quantum computation (AQC) for solving the three-satisfiability (3-SAT) problem. We have proposed a neuromorphic quantum computation algorithm based on AQC, in which an analogy to an artificial neural network (ANN) is considered in order to design a Hamiltonian. However, in the neuromorphic AQC, the relation between its computation time and success rate has not been clear. In this paper, we study residual energy and the probability of correct answers as a function of computation time. The residual energy behaves as expected from the adiabatic theorem. On the other hand, the success rate strongly depends on energy level crossings of excited states during Hamiltonian evolution. The results indicate that computation time must be adjusted according to a target problem.

1. Introduction

The number of computational steps increases exponentially when we try to solve a combinational optimization problem. If the size of a target problem is large, we cannot solve it in a reasonable computation time. It is expected that quantum computation can be viewed as a new information processing for solving such problems efficiently. Farhi *et al.* [1]-[3] have proposed an adiabatic quantum computation (AQC) for solving the three-satisfiability (3-SAT) problem. A database search can be done by using AQC in steps proportional to $O(\sqrt{N})$ [3], because AQC has an analog analogy to Grover's algorithm [4], [5]. In AQC, a quantum computation is done by changing a Hamiltonian enough slowly for fulfilling the requirement of the adiabatic theorem [6]. The quantum state stays always in each ground state during Hamiltonian evolution, so that decoherence effect is not severe in comparison with other quantum computation algorithms. The method of designing a final Hamiltonian for the original AQC has not been known except a few problems. In order to give a guideline to obtain a Hamiltonian, we have proposed a neuromorphic adiabatic quantum computation (neuromorphic AQC), in which an analogy to an artificial neural network (ANN [7]) is considered[8], [9]. The final Hamiltonian matrix is composed of diagonal and non-zero non-diagonal elements in which interactions between neurons are converted to interactions between qubits. This leads that a final quantum state is given as a superposition of solution and non-solution states. To evaluate the performance of neuromorphic AQC, it is necessary to study not only the residual energy but also the probability for finding correct answers. In this paper, first we study the relation between the residual energy and the computation time in neuromorphic AQC by numerical simulations. Next, the probability for finding correct answers is studied to clarify the performance more precisely.

2. Adiabatic Quantum Computation with Neuron-like Interconnections

2.1. Adiabatic Quantum Computation

The Hamiltonian of the adiabatic quantum computation changes as time goes on from an initial Hamiltonian H_I whose ground state is given as the superposition of all states, to a final Hamiltonian H_F whose ground state includes solutions for a given problem. (see [1], [8] for details.) The evolution of the Hamiltonian is given as

$$H(t) = \left(1 - \frac{t}{T}\right) H_I + \frac{t}{T} H_F, \tag{1}$$

$$H_I = \sum_{i=0}^{2^n-1} \sigma_x^{(i)}, \tag{2}$$

where n and $\sigma_x^{(i)}$ are the number of qubits and the x-component of the Pauli spin matrix acting on the i-th qubit, respectively. T denotes the computation time in which the Hamiltonian evolves. The initial quantum state is set to the superposition of all states and is given as

$$|\psi(0)\rangle = \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |i\rangle\,, \tag{3}$$

where $|i\rangle \equiv |x_n\rangle \cdots |x_i\rangle \cdots |x_2\rangle\,|x_1\rangle \equiv |x_n \cdots x_i \cdots x_2 x_1\rangle$ and each qubit $|x_i\rangle$ takes $|0\rangle$ or $|1\rangle$, exclusively. We can control the speed of state changes to be suitable for finding

solutions. If a sufficiently large T is chosen, the evolution becomes adiabatic. The adiabatic theorem [6] tells that a quantum state remains close to each ground state. Therefore, solutions can be found in the final state $|\psi(T)\rangle$. However, if there is any degeneracy or level crossing during the Hamiltonian evolution, the quantum state may not stay in a ground state. These behaviors depend on a Hamiltonian strongly.

2.2. Neuromorphic Method for Designing Final Hamiltonians

The method of designing a final Hamiltonian H_F has not been known for a general case. First of all, we consider a Hopfield neural network (HNN) [10]. The energy function E is defined for an HNN because it has symmetrical synaptic connections. It is given as

$$E = -\frac{1}{2}\sum_{i,j} w_{ij} o_i o_j, \tag{4}$$

where o_i and $w_{ij}(= w_{ji})$ are the output of the i-th neuron and the synaptic weight between the i-th and the j-th neurons, respectively. The network state changes to lower energy states as time goes on. It has been known that an HNN can be applied to a combinatorial optimization problem [10]. If we can obtain the synaptic weights by comparing the energy function with the cost or penalty function of an optimization problem, the HNN can work for solving such a problem. However, if the network is trapped at a local minimum, its final state does not correspond to a solution. If we know a method to convert a synaptic weight matrix W to a final Hamiltonian H_F, AQC can be applied to an optimization problem in which quantum dynamics must be helpful for avoiding the trap at local minima.

Table 1: Relation between Hamiltonian and synaptic weight

Hamiltonian	Ground State	Measured State	Interaction	Synaptic Weight
$\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & A & 0 \\ 0 & A & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$	$\|01\rangle - \|10\rangle$	$\|01\rangle, \|10\rangle$	inhibitory	$w_{ij} < 0$
$\begin{pmatrix} \lambda & 0 & 0 & A \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ A & 0 & 0 & \lambda \end{pmatrix}$	$\|00\rangle - \|11\rangle$	$\|00\rangle, \|11\rangle$	excitatory	$w_{ij} > 0$

Table I describes how to design a final Hamiltonian [8], [9]. As for the upper Hamiltonian, its ground state is composed of $|01\rangle$ and $|10\rangle$ states due to the existence of non-diagonal elements A. When the quantum system is in a ground state, possible states to be measured is $|01\rangle$ or $|10\rangle$, exclusively. It can be said that the interaction of two neurons is inhibitory if we consider the analogy with an ANN model. Also, excitatory interaction is possible with the lower Hamiltonian. In order to relate a synaptic weight with a qubit interaction, the upper (lower) Hamiltonian is used for a negative (positive) synaptic weight. The sum of the upper and lower Hamiltonians reflecting a synaptic matrix W is employed as a final Hamiltonian of neuromorphic AQC. Please note that these Hamiltonians can be replaced by some suitable ones arbitrary. Both parameters λ and A are constants, and their magnitudes should reflect the value of a synaptic weight w_{ij}. In the followings, however, we do not pay much attention to the magnitude of there parameters because the synaptic weights in this paper take values 0 or -1 only. For example, suppose we have a synaptic weight matrix

$$W_1 = \begin{pmatrix} 0 & -1 & -1 & 1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ 1 & -1 & -1 & 0 \end{pmatrix}. \tag{5}$$

This matrix represents one excitatory and five inhibitory interactions. Then a final Hamiltonian H_{F1} is given according to the rules described in Table I as

$$H_{F1} = \begin{pmatrix}
\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda & A & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & A & \lambda & 0 & A & 0 & 0 & 0 & A & 0 & 0 & A & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda & 0 & A & A & 0 & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & A & A & 0 & \lambda & 0 & 0 & 0 & A & 0 & 0 & 0 & 0 & A & 0 & 0 \\
0 & 0 & 0 & A & 0 & \lambda & A & 0 & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & A & 0 & A & \lambda & 0 & 0 & 0 & A & 0 & A & 0 & 0 & A \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & A & 0 & A & 0 & 0 \\
0 & 0 & A & 0 & A & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
A & 0 & 0 & A & 0 & A & 0 & 0 & 0 & \lambda & A & 0 & A & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A & 0 & 0 & A & \lambda & 0 & A & 0 & 0 & 0 \\
0 & 0 & A & 0 & 0 & 0 & 0 & A & 0 & 0 & 0 & \lambda & 0 & A & A & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A & 0 & 0 & A & A & 0 & \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A & 0 & 0 & A & 0 & 0 & 0 & A & 0 & \lambda & A & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A & 0 & \lambda & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda
\end{pmatrix}, \tag{6}$$

where λ and A are arbitrary constants.

3. Simulation Results

3.1. Residual Energy

For discussing the performance of the neuromorphic AQC, we investigate residual energy ΔE. It is defined as the energy difference between the nominal ground energy calculated from a final Hamiltonian and the actual energy after the state evolution, and given as

$$\Delta E = \langle \psi^*(T) \mid H_F \mid \psi^*(T) \rangle - \langle \psi(T) \mid H_F \mid \psi(T) \rangle, \tag{7}$$

where $|\psi^*(T)\rangle$ represents a theoretical final state. If there is no degeneracy in the ground energy level of H_F, $|\psi^*(T)\rangle$ is identical with $|\psi_g(H_F)\rangle$, which corresponds to the single ground state obtained analytically. On the other hand, for the case H_F has degenerated ground states, we define $|\psi^*(T)\rangle$ as a liner combination of the ground states $|\psi_g^i(H_F)\rangle$, namely

$$|\psi^*(T)\rangle = \sum_i C_g^i \, |\psi_g^i(H_F)\rangle. \tag{8}$$

C_g^i is the probability amplitude of the ground state $|\psi_g^i(H_F)\rangle$ where i denotes the index of the degenerated ground states. The normalization condition

$$\sum_i \mid C_g^i \mid^2 = 1 \tag{9}$$

gives an obvious result

$$\langle \psi^*(T) \mid H_F \mid \psi^*(T) \rangle = \langle \psi_g^i(H_F) \mid H_F \mid \psi_g^i(H_F) \rangle \qquad (10)$$

for all i. Therefore, ΔE does not depend on the choice of C_g^is.

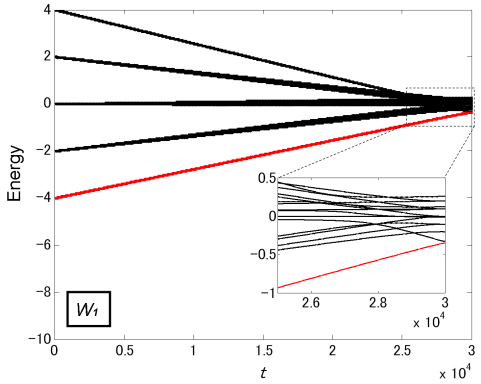

Figure 1: Energy change of a 4-qubit system with the Hamiltonian H_{F1}. Red line denotes each instantaneous ground state.

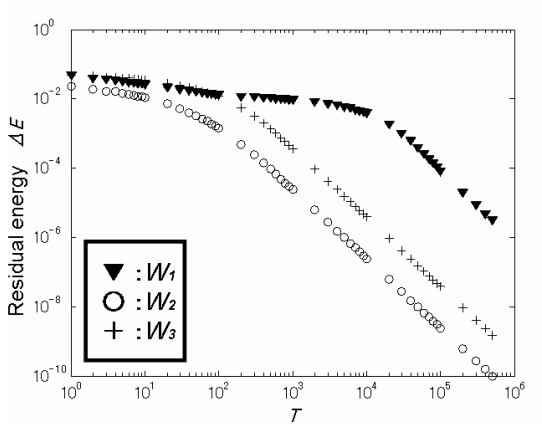

Figure 2: Change of the residual energy ΔE for three different Hamiltonians as a function of the computation time T. (▼:W_1,○:W_2,+:W_3).

If there is no degeneracy or level crossing in the ground state, the relation

$$\Delta E \propto O\left(\frac{1}{T^2}\right) \qquad (11)$$

in the long T limit should be found as expected from the adiabatic theorem. Figure 1 shows the energy change of a 4-qubit system during the Hamiltonian evolution from H_I to H_{F1}, where H_{F1} is obtained from W_1. It can be seen that there is no degeneracy or level crossing, so that an adiabatic change is realized. Also, the residual energy shows $O(1/T^2)$ dependence as shown in Fig.2. It agrees well with the results for quantum annealing reported by S. Suzuki[11]. Also, the results for additional two 3-qubit systems have been shown in the figure. Their qubit interactions are obtained from synaptic weight matrices

$$W_2 = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}, \qquad (12)$$

and

$$W_3 = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}, \qquad (13)$$

respectively.

3.2. Probability of Correct Answers

In the previous section, it has been confirmed that the neuromorphic AQC shows a similar behavior as the original AQC in terms of the residual energy. The investigation only of the residual energy is not sufficient for performance evaluation of the neuromorphic AQC because both solution and error states are often mixed together in a ground state of H_F. Namely, one of ground states at $t = T$ is given as

$$|\psi_g^i(H_F)\rangle = \sum_{j\in\Omega} C_j^i |j\rangle + \sum_{k\notin\Omega} C_k^i |k\rangle, \qquad (14)$$

where Ω denotes a set of solution states. In general, this mixing is not avoidable because a final Hamiltonian is obtained from qubit-qubit interactions. Therefore, it is necessary to evaluate successful rates as a function of computaiton time T apart from the residual energy.

First, we define the probability difference as

$$\begin{aligned} \Delta P_g &\equiv 1 - P^* \\ &= 1 - |\langle \psi(T) \mid \psi^* \rangle|^2 \\ &= 1 - \sum_i |C_g^i|^2 |\langle \psi(T) \mid \psi_g^i(H_F) \rangle|^2 \end{aligned} \qquad (15)$$

where P^* is the actual probability of the quantum state $|\psi(T)\rangle$ found in the merged ground state $|\psi^*(T)\rangle$ in Eq.(8). Figure 3(a) shows the change of ΔP_g for the final Hamiltonians H_{F1}, H_{F2} and H_{F3}. When we assume $\Delta P_g \propto O(1/T^\zeta)$, $\zeta \sim 2$ is obtained again. However, both solution and error states are mixed together in the final quantum state $|\psi(T)\rangle$. We should investigate probabilities of solution states, and define a probability difference given as

$$\begin{aligned} \Delta P_s &\equiv |\sum_{j\in\Omega} P_j^* - P_j| \\ &= ||\sum_{j\in\Omega} \langle j \mid \psi^*(T)\rangle|^2 - |\sum_{j\in\Omega} \langle j \mid \psi(T)\rangle|^2|, \end{aligned} \qquad (16)$$

where P_j is the actual probability of the solution state $|j\rangle$ found in the final quantum state $|\psi(T)\rangle$, and P_j^* is the theoretical probability given from the final Hamiltonian H_F.

Figure 3(b) shows the change of ΔP_s for the final Hamiltonians H_{F1}, H_{F2} and H_{F3}. $\zeta \sim 2$ is obtained for W_1 and W_3. On the other hand, $\zeta \sim 1$ is obtained for W_2 when T is large enough. Therefore, we can suppose that ζ depends on a final Hamiltonian H_F. This difference is related to the behavior of excited states. For H_{F2} the first and second excited states are exchanged during Hamiltonian evolution as shown in Fig.4. On the other hand, for H_{F1} and H_{F3} such exchange is not found. Not large but finite population changes in these excited states seem to affect to the performance significantly.

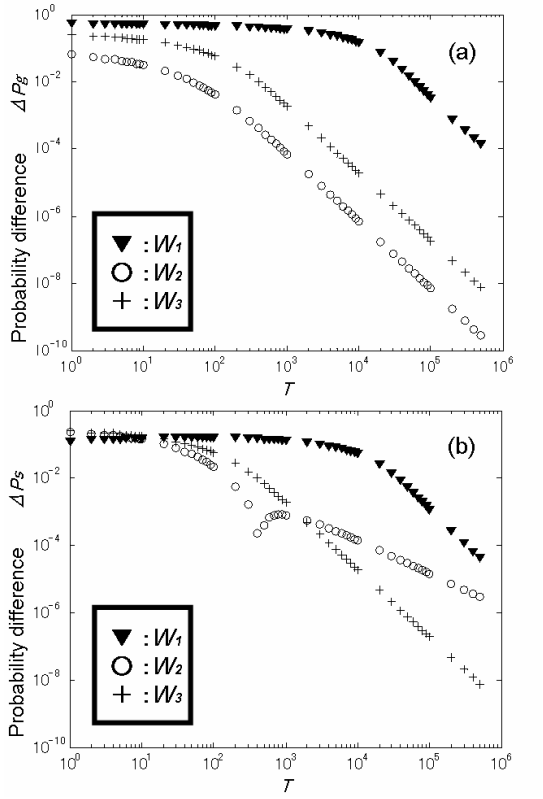

Figure 3: The probability differences ΔP_g (a) and ΔP_s (b) as a function of the computation time T. (▼:W_1,○:W_2,+:W_3).

4. Conclusion

In order to evaluate the performance of AQC with neuron-like interactions (neuromorphic AQC), we have studied the residual energy and the probability of correct answers as a function of computation time T. Unlike the original AQC, the final state is composed of solution and error states in the neuromorphic AQC. Therefore, the performance of neuromorphic AQC can not be evaluated only by a relation between residual energy and the computation time. It has been confirmed that the residual energy change is given as $O(1/T^2)$, and the ΔP_s change is given as

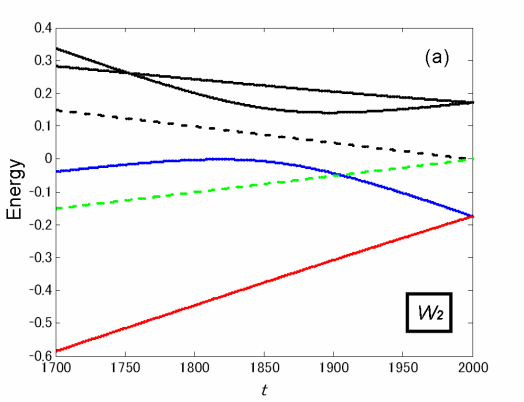

Figure 4: Energy change of a 3-qubit system with the Hamiltonian H_{F2}. Red, blue and green curves represent the ground, the first excited and the second excited states, respectively.

$O(1/T^2)$ or $O(1/T)$. The success rate depends on the characteristics or structure of a final Hamiltonian. The results indicate that computation time must be adjusted according to a target problem.

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