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Generation Method of Extremely Ill-conditioned Integer Matrices

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Abstract—This paper proposes an innovative generation method of extremely ill-conditioned integer matrices. This method is superior to the conventional Rump's method, i.e., the former has a simpler algorithm and can generate more variety of ill-conditioned matrices than the latter. ∴

1. Introduction

Extremely ill-conditioned matrices are required to examine the quality of accuracy-guaranteed algorithms for solving linear simultaneous equations[1]–[6]. Here an ill-conditioned matrix implies that its condition number is $10^{16} \sim 10^{100}$ or larger in the double precision arithmetic.

Once S. Rump[7] proposed a method to generate extremely ill-conditioned matrices. His method utilized *the Pell equation*, which is well-known in the number theory[12]. The method is most well-known and is used as a standard tool to generate an ill-conditioned matrix with an arbitrary condition number in the INTLAB, but the variety of generated matrices is not so large because the number of solutions of the Pell equation is not so many. So we want other methods to obtain more variety of matrices. From this point of view we proposed [8][10][11] several methods which are considered as extensions of Rump's method[7].

In this paper we propose another method to generate ill-conditioned matrices. It has the following features in comparison with Rump's method[7] and its extensions [8][10][11]: (i) it has a simpler algorithm, (ii) the obtainable condition number is roughly the same as previous ones, (iii) it generates much variety of matrices. The obtained matrices are somewhat similar to the companion matrix.

2. Preliminaries

Let μ be a large positive integer such as 10^8, 10^{16} or 2^{53} (but $\mu = 10$ or $\mu = 2$ may also be permissible theoretically) and let an $n \times n$ integer matrix $A = [a_{ij}]$ to be determined satisfy $|a_{ij}| \leq \mu$. It is very probable that the maximum condition number of A is large, as μ is large. Our purpose is to generate an integer matrix $A = [a_{ij}]$ such that $|a_{ij}| \leq \mu$ and $\mathrm{Cond}(A)(= \|A\|\|A^{-1}\|)$ is extremely large.

2.1. Outline of Rump's method

One of the key points of Rump's method is to find a 2×2 integer matrix V[1] s.t.

$$V = \begin{bmatrix} P & kQ \\ Q & P \end{bmatrix}, \quad |V| = \begin{vmatrix} P & kQ \\ Q & P \end{vmatrix} = 1 \tag{1}$$

The condition $|V| = 1$ is important for an ill-conditioned matrix. The integers P and Q are extremely large such as 10^{50} and are chosen so as to satisfy the Pell equation:

$$P^2 - kQ^2 = 1 \tag{2}$$

Thus $|V| = 1$ in Eq.(1) is satisfied. Then utilizing V in Eq.(1), he proposed a $(2n + 2) \times (2n + 2)$ integer matrix A and showed by rather tricky calculations that A satisfies

$$\mathrm{Cond}_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty \geq (P + Q)(P + kQ) \sim 4\mu^{2(n+1)} \tag{3}$$

The last term $(4\mu^{2(n+1)})$ is obtained under certain reasonable assumptions. Since A is a $(2n + 1) \times (2n + 1)$ matrix, we see that the condition number in Eq.(3) per degree is approximately μ, i.e., $\{\mathrm{Cond}_\infty(A)\}^{1/(2n+1)} \approx \mu$.

2.2. Previous extensions of Rump's method

We showed[8][10][11] that Rump's algorithm can easily be generalized by replacing V in Eq.(1) with the following two kinds of matrices.

2.2.1 Replacing V by more general type of a 2×2 matrix

The matrix V was generalized as:

$$V' = \begin{bmatrix} P & F \\ Q & G \end{bmatrix}, \quad |V'| = PG - QF = 1 \tag{4}$$

Prescribed P and Q having no common factor, e.g.,

$$\begin{aligned} P &= 2^k, \quad Q = 3^m \\ P &= 2^{k1}5^{k2}11^{k3}, \quad Q = 3^{m1}7^{m2} \end{aligned}$$

we can find F and G satisfying Eq.(4) by using the Euclid algorithm[13].

2.2.2 Replacing V by a 3×3 matrix

3. Generation of ill-conditioned matrices similar to the companion matrix

3.1. Generation method

In this section we consider the generation of an ill-conditioned matrix, which is similar to a companion ma-

[1] The symbol V is different from that in the original Rump's paper[7].

trix. Let A be an $n \times n$ integer matrix such that

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & \cdots & a_{n-1} & a_n \\ 1 & -\sigma_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -\sigma_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -\sigma_3 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & -\sigma_{n-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -\sigma_{n-1} \end{bmatrix} \tag{5}$$

Without loss of generality we can assume

$$0 < \sigma_i < \mu \quad (i = 1, 2, \cdots, n-1) \tag{6}$$

$$|a_i| < \mu \quad (i = 1, 2, \cdots, n) \tag{7}$$

In this paper we determine a_i $(i = 1, \cdots, n)$ such that

$$(((a_1\sigma_1 + a_2)\sigma_2 + a_3)\sigma_3 + \cdots)\sigma_{n-1} + a_n = 1 \tag{8}$$

Eq.(8) corresponds to $|V| = 1$ in Eq.(1). From the above we see that a_i $(i = 1, 2, \cdots, n)$ necessarily include both positive and negative values. Referring to Eq.(8), we will describe how we determine a_i.

3.2. Determination of a_i

We determine a_i by the following three steps:
Step 1: From Eq.(8) we have

$$1 - a_n \equiv 0 \pmod{\sigma_{n-1}} \tag{9}$$

from which we have

$$\frac{1 - a_n}{\sigma_{n-1}} =: k_{n-1}, \quad (k_{n-1} = 0, \pm 1, \pm 2, \pm 3, \cdots) \tag{10}$$

Thus we have

$$a_n = 1 - \sigma_{n-1}k_{n-1} \tag{11}$$

Therefore k_{n-1} has to satisfy from Eq.(7) the following equation:

$$|a_n| = |1 - \sigma_{n-1}k_{n-1}| < \mu \tag{12}$$

i.e., $-\mu < 1 - \sigma_{n-1}k_{n-1} < \mu$. This can be rewritten as

$$1 + \mu > \sigma_{n-1}k_{n-1} > -\mu + 1 \tag{13}$$

from which we have

$$\frac{1+\mu}{\sigma_{n-1}} > k_{n-1} > \frac{-\mu + 1}{\sigma_{n-1}} \tag{14}$$

From the above and Eq.(14) we see that

$$k_{n-1} = \left[\frac{1+\mu}{2\sigma_{n-1}}\right](>0) \ \text{ or } \ \left[\frac{1-\mu}{2\sigma_{n-1}}\right](<0) \tag{15}$$

are appropriate candidate of k_{n-1}, but k_{n-1} is not limited to Eq.(15). We have a_n from k_{n-1} and Eq.(11) by Eq.(11).
Step 2: Quite similarly we can derive equations corresponding to Eqs.(9)–(15) successively. We take $j = 1, 2, \cdots, n-1$ in the order. Then we have $k_{n-2}, k_{n-3}, \cdots, k_1, a_{n-1}, a_{n-2}, \cdots, a_2$ as follows:

$$k_{n-j} - a_{n-j} \equiv 0 \pmod{\sigma_{n-j-1}} \tag{16}$$

$$\frac{k_{n-j} - a_{n-j}}{\sigma_{n-j-1}} \equiv k_{n-j-1}, \quad (k_{n-j-1} = 0, \pm 1, \pm 2, \cdots) \tag{17}$$

$$a_{n-j} = k_{n-j} - \sigma_{n-j-1}k_{n-j-1} \tag{18}$$

$$|a_{n-j}| = |k_{n-j} - \sigma_{n-j-1}k_{n-j-1}| < \mu \tag{19}$$

$$\frac{k_{n-j}+\mu}{\sigma_{n-j-1}} > k_{n-j-1} > \frac{-\mu + k_{n-j}}{\sigma_{n-j-1}} \tag{20}$$

We therefore have:

$$k_{n-j-1} = \left[\frac{k_{n-j}+\mu}{2\sigma_{n-j-1}}\right](>0) \text{ or } \left[\frac{-\mu + k_{n-j}}{2\sigma_{n-j-1}}\right](<0) \tag{21}$$

are appropriate candidate of k_i.
Step 3:

$$a_1 \equiv k_1 \tag{22}$$

Since a_i have to satisfy Eq.(8), we assume
Assumption 1: We choose a_i as

$a_{2i} > 0,\ a_{2i+1} < 0 \quad \therefore k_{2i} < 0,\ k_{2i+1} > 0\ (i = 1, 2, \cdots,)$
or
$a_{2i} < 0,\ a_{2i+1} > 0 \quad \therefore k_{2i} > 0,\ k_{2i+1} < 0\ (i = 1, 2, \cdots,)$

For convenience let

$$k_n = 1 \tag{23}$$

Then we can calculate a_j and k_j recursively for $j = n-1, n-2, \cdots, 2$ in this order by both Eq.(24) and Assumption 1.

$$\frac{k_{j+1}+\mu}{\sigma_j} > k_j > \frac{-\mu + k_{j+1}}{\sigma_j} \tag{24}$$

$$k_j = \left[\frac{k_{j+1}+\mu}{2\sigma_j}\right](>0) \ \text{ or } \ \left[\frac{k_{j+1}-\mu}{2\sigma_j}\right](<0) \tag{25}$$

is reasonable candidates of k_j.

$$a_{j+1} = k_{j+1} - \sigma_j k_j \tag{26}$$

Eq.(25) is only an example of choice and we can also choose many other values.

3.3. Condition number of A in Eq.(5)

In this section we calculate the condition number of A in Eq.(5) with the ∞-norm. For this purpose we need to calculate the inverse matrix A^{-1}, which can be easily calculated in a similar way as in [8] as follows:
Let

$$H \equiv \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & \prod_1^{n-1}\sigma_i \\ 0 & 1 & 0 & 0 & \cdots & 0 & \prod_2^{n-1}\sigma_i \\ 0 & 0 & 1 & 0 & \cdots & 0 & \prod_3^{n-1}\sigma_i \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & \sigma_{n-2}\sigma_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \sigma_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \tag{27}$$

Then we have

$$\begin{aligned} A' &\equiv AH \\ &= \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & \cdots & a_{n-2} & a_{n-1} & a_n \\ 1 & -\sigma_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -\sigma_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -\sigma_3 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -\sigma_{n-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -\sigma_{n-1} \end{bmatrix} \\ &\times \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \prod_1^{n-1}\sigma_i \\ 0 & 1 & 0 & \cdots & 0 & 0 & \prod_2^{n-1}\sigma_i \\ 0 & 0 & 1 & \cdots & 0 & 0 & \prod_3^{n-1}\sigma_i \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & \sigma_{n-2}\sigma_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & 1 & \sigma_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$$
= \left[\begin{array}{ccccccc|c}
a_1 & a_2 & a_3 & a_4 & \cdots & a_{n-2} & a_{n-1} & 1 \\ \hline
1 & -\sigma_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & -\sigma_2 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & -\sigma_3 & \cdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & 1 & -\sigma_{n-2} & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0
\end{array}\right]
$$

Let

$$
\begin{aligned}
A' &= \left[\begin{array}{c|c} U & 1 \\ \hline W & 0 \end{array}\right] \\
U &= \left[\begin{array}{ccccc} a_1 & a_2 & a_3 & \cdots & a_{n-1} \end{array}\right] \\
W &= \left[\begin{array}{cccccc}
1 & -\sigma_1 & 0 & 0 & \cdots & 0 \\
0 & 1 & -\sigma_2 & 0 & \cdots & 0 \\
0 & 0 & 1 & -\sigma_3 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 1 & -\sigma_{n-2} \\
0 & 0 & 0 & \cdots & 0 & 1
\end{array}\right]
\end{aligned}
$$

$$
\therefore \quad (A')^{-1} = \left[\begin{array}{cc} 0 & W^{-1} \\ 1 & -UW^{-1} \end{array}\right]
$$

$$
\begin{aligned}
W^{-1} &= \left[\begin{array}{cccccc}
1 & \sigma_1 & \sigma_1\sigma_2 & \prod_1^3 \sigma_i & \cdots & \prod_1^{n-2} \sigma_i \\
0 & 1 & \sigma_2 & \sigma_2\sigma_3 & \cdots & \prod_2^{n-2} \sigma_i \\
0 & 0 & 1 & \sigma_3 & \cdots & \prod_3^{n-2} \sigma_i \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 1 & \sigma_{n-2} \\
0 & 0 & 0 & \cdots & 0 & 1
\end{array}\right] \\
-UW^{-1} &= -\left[\begin{array}{ccccc} a_1 & a_2 & a_3 & \cdots & a_{n-1} \end{array}\right] \\
&\times \left[\begin{array}{cccccc}
1 & \sigma_1 & \sigma_1\sigma_2 & \prod_1^3 \sigma_i & \cdots & \prod_1^{n-2} \sigma_i \\
0 & 1 & \sigma_2 & \sigma_2\sigma_3 & \cdots & \prod_2^{n-2} \sigma_i \\
0 & 0 & 1 & \sigma_3 & \cdots & \prod_3^{n-2} \sigma_i \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 1 & \sigma_{n-2} \\
0 & 0 & 0 & \cdots & 0 & 1
\end{array}\right] \\
&= -\left[\begin{array}{ccccc} K_1, & K_2, & K_3, & \cdots, & K_{n-1} \end{array}\right]
\end{aligned}
$$

where

$$
K_j \equiv a_1 \prod_{i=1}^{j-1} \sigma_i + a_2 \prod_{i=2}^{j-1} \sigma_i + \cdots + a_j \quad (j = 1, 2, \cdots, n-1)
$$

i.e.,

$$
\begin{aligned}
K_1 &= a_1 \\
K_2 &= a_1\sigma_1 + a_2 \\
K_3 &= a_1\sigma_1\sigma_2 + a_2\sigma_2 + a_3 \\
&\vdots \\
K_{n-1} &= a_1 \prod_{i=1}^{n-2} \sigma_i + \cdots
\end{aligned}
$$

So we have the final form of A^{-1} as:

$$
\begin{aligned}
A^{-1} &= H(A')^{-1} \qquad (28) \\
&= \left[\begin{array}{ccccccc}
1 & 0 & 0 & 0 & \cdots & 0 & \prod_1^{n-1} \sigma_i \\
0 & 1 & 0 & 0 & \cdots & 0 & \prod_2^{n-1} \sigma_i \\
0 & 0 & 1 & 0 & \cdots & 0 & \prod_3^{n-1} \sigma_i \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & 1 & \sigma_{n-2}\sigma_{n-1} \\
0 & 0 & 0 & 0 & \cdots & 0 & \sigma_{n-1} \\
0 & 0 & 0 & 0 & \cdots & 0 & 1
\end{array}\right] \\
&\times \left[\begin{array}{ccccccc}
0 & 1 & \sigma_1 & \sigma_1\sigma_2 & \prod_1^3 \sigma_i & \cdots & \prod_1^{n-2} \sigma_i \\
0 & 0 & 1 & \sigma_2 & \sigma_2\sigma_3 & \cdots & \prod_2^{n-2} \sigma_i \\
0 & 0 & 0 & 1 & \sigma_3 & \cdots & \prod_3^{n-2} \sigma_i \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & 1 & \sigma_{n-2} \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 \\
1 & -K_1 & -K_2 & -K_3 & \cdots & \cdots & -K_{n-1}
\end{array}\right] \\
&= \left[\begin{array}{cccc}
\prod_1^{n-1} \sigma_i & 1 - K_1 \prod_1^{n-1} \sigma_i & \cdots & \prod_1^{n-2} \sigma_i - K_{n-1} \prod_1^{n-1} \sigma_i \\
\prod_2^{n-1} \sigma_i & -K_1 \prod_2^{n-1} \sigma_i & \cdots & \prod_2^{n-2} \sigma_i - K_{n-1} \prod_2^{n-1} \sigma_i \\
\cdots\cdots & \cdots & \cdots & \\
1 & -K_1 & \cdots & -K_{n-1}
\end{array}\right]
\end{aligned}
$$

We therefore have

$$
\|A^{-1}\|_\infty > \max\left\{ \prod_1^{n-1} \sigma_i, (-K\sigma_{n-1} + 1) \prod_{i=1}^{n-2} \sigma_i \right\} \quad (29)
$$

$$
\|A\|_\infty > \max\left\{ \sum_{i=1}^{n} |a_i|, \max(\sigma_i + 1) \right\} \quad (30)
$$

Finally we have

$$
\mathrm{Cond}_\infty(A) > \left(\prod_1^{n-1} \sigma_i \right) \sum_1^n |a_i| \quad (31)
$$

This corresponds to the Rump's result in Eq.(3). If we choose

$$
|a_i| \sim \mu, \quad \nu_i \sim \sqrt{\mu} \quad (32)
$$

then we see that $K_{n-1} \sim \mu^{\frac{n}{2}-1}$. We therefore see from Eqs.(31) and (32) that

$$
\mathrm{Cond}_\infty(A) > \mu^{\frac{n}{2}-1} \cdot \mu^{\frac{n-1}{2}} (n-1)\mu \sim n\mu^{n-\frac{1}{2}} \quad (33)
$$

Since the size of A is n, the condition number per degree is approximately μ, that is, Eq.(31) is approximately equal to that in [7].

3.4. Considerations through examples

Example 1: Let

$$
\mu = 10, \quad n = 4, \quad \sigma_1 = \sigma_2 = \sigma_3 = 5 \quad (34)
$$

We choose k_i and a_i using Eqs.(9)– (22).

Since $1 - a_4$ must be divided by 5, we have

$$
a_4 = 1, -4, 6, -9, \cdots
$$

So we choose $a_4 = -9$ as an example. Then $k_3 = (1 - a_4)/5 = 2$.

Since $k_3 - a_3$ must be divided by 5, we choose $a_3 = 7$ and therefore $k_2 = -1$. Since $(k_2 - a_2)$ must be divided by 5, we choose as $a_2 = -6$ and $k_1 = a_1 = 1$. Then $((1 \times 5 + (-6)) \times 5 + 7) \times 5 + (-9)) = 1$ surely holds.

We therefore have

$$
A = \left[\begin{array}{cccc}
1 & -6 & 7 & -9 \\
1 & -5 & 0 & 0 \\
0 & 1 & -5 & 0 \\
0 & 0 & 1 & -5
\end{array}\right] \quad (35)
$$

$$
A^{-1} = \left[\begin{array}{cccc}
125 & -124 & 130 & -225 \\
25 & -25 & 26 & -45 \\
5 & -5 & 5 & -9 \\
1 & -1 & 1 & -2
\end{array}\right] \quad (36)
$$

from which we see that

$$\mathrm{Cond}_\infty(A) = \|A\| \cdot \|A^{-1}\| = 13892 \tag{37}$$

We have $|A| = -1$ and the singular values of A are about 14.1, 5.14, 4.42, and 0.00312. Therefore we have

$$\mathrm{Cond}_2(A) = \frac{14.109418}{0.0031213} \approx 4520.2995$$

Remark 1: From the above results, we see that (i) A has a considerably large condition number even for such small μ and n, (ii) A has three large singular values and an extremely small one, and (iii) A^{-1} is very near to a matrix with rank one.

Example 2: Let

$$\mu = 1000, \quad n = 4, \quad \sigma_1 = \sigma_2 = \sigma_3 = 50 \tag{38}$$

In a similar way as in Example 1 we choose k_i and a_i using Eqs.(9)–(22).

Since $1-a_4$ must be divided by 50, we choose $a_4 = -799$ and $k_3 = (1 - a_4)/50 = 16$.

Since $k_3 - a_3$ must be divided by 50, we choose as $a_3 = 716$ and therefore $k_2 = -14$. Since (k_2-a_2) must be divided by 50, we choose as $a_2 = -864$ and $k_1 = a_1 = 17$. Then $((17 \times 50 + (-864)) \times 50 + 716) \times 50 + (-799)) = 1$ holds.

We therefore have

$$A = \begin{bmatrix} 17 & -864 & 716 & -799 \\ 1 & -50 & 0 & 0 \\ 0 & 1 & -50 & 0 \\ 0 & 0 & 1 & -50 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 125000 & -2124999 & 1750050 & -1997500 \\ 2500 & -42500 & 35001 & -39950 \\ 50 & -850 & 700 & -799 \\ 1 & -17 & 14 & -16 \end{bmatrix}$$

from which we see that

$$\mathrm{Cond}_\infty(A) = \|A\| \cdot \|A^{-1}\| \approx 13 \times 10^8 \tag{39}$$

We have $|A| = -1$, and singular values are about 1380, 50.0, 49.4 and 0.0000003. Then we have

$$\mathrm{Cond}_2(A) = \frac{1378.5521}{0.0000003} \approx 4.693 \cdot 10^9$$

Thus A has a considerably large condition number for $\mu = 1000$ and $n = 4$ and a similar remark as Remark 1 in Example 1 holds.

We see from the above examples that we do not obtain uniform singular values distribution. The reason is omitted here due to the lack of space. As a trial to obtain more uniform singular value distribution, we can choose $\sigma_1 = 1$ but the result is not necessarily good. The desirable singular value distribution will be discussed in near future.

4. Conclusion

This paper proposed an innovative generation method of extremely ill-conditioned integer matrices. This method is superior to the conventional Rump's method in some respects.

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