Uncovering network architecture from controlled steady state responses

Dongchuan Yu†‡, and Martin Hasler‡

†College of Automation Engineering, Qingdao University, Qingdao, Shandong 266071, China
‡Laboratory of Nonlinear Systems, Ecole Polytechnique Federale de Lausanne (EPFL), CH-1015 Lausanne, Switzerland
Email: dongchuanyu@yahoo.com, martin.hasler@epfl.ch

Abstract—We suggest an architecture (i.e., node dynamics, connection topology, and coupling functions) identification method using pinning control to dynamical networks with unknown time-varying interaction-delays. The architecture identification method is illustrated with a cellular neural network.

1. Introduction

The research on complex networks of interacting dynamical systems [1] has rapidly attracted increasing interest in emerging cooperative phenomena of various real networks such as neurons, interacting genes, power stations, or coupled nonlinear oscillators. Current studies in this filed focused on how the topology properties of the network (such as clustering coefficient, connectivity distribution, and average network distance) influence the cooperative dynamic behavior (e.g., network synchronization) [2]. However the inverse problem – how to uncover network architecture (NA) (including node dynamics, connection topology, and coupling functions) from the dynamic evolution – has not been well understood [3] and is very important for analysis of real networks. In this paper, we show that controlled network steady state responses can be applied to identify NA of dynamical networks. Our previous method [3] is not applicable to estimate node dynamics and coupling functions and can be used for detecting only the connection topology of networks under the assumption that the node dynamics and coupling functions are known precisely.

We analyze a network of interacting dynamical systems,

$$\dot{\boldsymbol{x}}_i = \boldsymbol{f}_i(\boldsymbol{x}_i) + \sum_{j\in V, j\neq i} a_{ij}[\boldsymbol{h}_j(\boldsymbol{x}_{j,\tau_{ij}}) - \boldsymbol{h}_i(\boldsymbol{x}_i)] \qquad (1)$$

where $i \in V$ ($V := \{1, 2, \ldots, n\}$ is the set of *vertices*), $\boldsymbol{x}_i = [x_i, y_i, \ldots]^T \in \boldsymbol{R}^N$ is the state vector of node i, $\boldsymbol{f}_i : \boldsymbol{R}^N \to \boldsymbol{R}^N$ describes the local dynamics of node i, and $\boldsymbol{h}_j : \boldsymbol{R}^N \to \boldsymbol{R}^N$ is the coupling function of node j. For generality we also include some interaction delay $\tau_{ij}(t)$ which is a function of time and concerns the coupling from node j to i, with $\boldsymbol{x}_{j,\tau_{ij}}(t) := \boldsymbol{x}_j(t - \tau_{ij})$. The parameters a_{ij} are elements of the adjacency matrix $A = (a_{ij})$ describing the topology of the network connections (with $a_{ij} = 1$ if the node j is driving node i, and $a_{ij} = 0$ otherwise). The pinning control problem of the network is described as

$$\dot{\boldsymbol{x}}_i = \boldsymbol{f}_i(\boldsymbol{x}_i) + \sum_{j\in V, j\neq i} a_{ij}[\boldsymbol{h}_j(\boldsymbol{x}_{j,\tau_{ij}}) - \boldsymbol{h}_i(\boldsymbol{x}_i)] + \boldsymbol{u}_i \qquad (2)$$

where $i \in V$ and $\boldsymbol{u}_i$ are control signals to be designed.

Previous pinning control methods (cf. [4, 5]) are not applicable to identify the NA and always assume that: i) all nodes are identical (that is, $\boldsymbol{f}_i = \boldsymbol{f}_j = \boldsymbol{f}$ for all i, j) and without interaction-delay (that is, $\tau_{ij} = 0$ for all i, j). However, in this paper, we suggest a new form $\boldsymbol{u}_i = -\theta_i(\boldsymbol{x}_i - \hat{x}_{is})$ for the control signal. By choosing large enough gains θ_i, the network is driven to a steady state that is determined by θ_i and $\hat{x}_{is}$. Interestingly, we found that when proper $\hat{x}_{is}$ are chosen, the controlled steady state responses can be applied for uncovering the NA of the network.

2. Theory

For simplicity we consider a network of 1D oscillators with scalar states x_i and local node dynamics $f_i : \boldsymbol{R} \to \boldsymbol{R}$. We assume that $f_i(0) = 0$ and $h_i(0) = 0$ for all i; and the mappings f_i and h_i are Lipschitzian, that is, there exist positive constants L_{1i} and L_{2i} such that $|f_i(y_i) - f_i(x_i)| \leq L_{1i}|y_i - x_i|$ and $|h_i(y_i) - h_i(x_i)| \leq L_{2i}|y_i - x_i|$, for all i. Furthermore we assume for the *unknown* interaction delays τ_{ij} that there exist positive T_{dij} and δ_{ij} such that $0 \leq \tau_{ij}(t) \leq T_{dij}$ and $\dot{\tau}_{ij}(t) \leq \delta_{ij} < 1$, for all i, j. The pinning control problem of the network is given by

$$\dot{x}_i = f_i(x_i) + \sum_{j\in V, j\neq i} a_{ij}[h_j(x_{j,\tau_{ij}}) - h_i(x_i)] + u_i, \qquad (3)$$

where u_i are control signals to be designed.

2.1. Driving the network to steady-state

The following theorem (its proof is omitted for compactness) gives the rules to design the control signal for driving the network to steady-states.

Theorem 1 – When u_i has the form

$$u_i = -\theta_i(x_i - \hat{x}_{is}), \qquad (4)$$

with constant $\hat{x}_{is}$ freely chosen and

$$\theta_i > \max[\gamma_i, \rho_i] \qquad (5)$$

wherein $\gamma_i := L_{1i} + D_i^{in} + \frac{1}{2}D_i^{in}L_{2i}^2 + \frac{D_i^{out}}{2(1-\delta_{ij}^{max})}L_{2i}^2$ and $\rho_i := 4L_{1i} + (4D_i^{in} + 2D_i^{out})L_{2i} + 2\sum_{j\in V, j\neq i} a_{ij}L_{2j}$, the network (3) is driven to steady states $(x_{1s}, x_{2s}, \ldots, x_{ns})$, satisfying

$$\sum_{j\in V, j\neq i} a_{ij}[h_j(x_{js}) - h_i(x_{is})] = \theta_i(x_{is} - \hat{x}_{is}) - f_i(x_{is}), \forall i \in V \quad (6)$$

and

$$|x_{is}| \le \sqrt{\frac{2\theta_{max}}{\theta_{min}} \sum_{k\in V} \hat{x}_{ks}^2}, \qquad \forall i \in V \tag{7}$$

with $D_i^{in} := \sum_{k\in V, k\neq i} a_{ik}$, $D_i^{out} := \sum_{k\in V, k\neq i} a_{ki}$, $\delta_{ij}^{max} := \max_{i,j}\delta_{ij}$, $\theta_{min} := \min_k \theta_k$, and $\theta_{max} := \max_k \theta_k$.

2.2. Detecting network connection topology

We now show how to use the steady state equation set (6) in Theorem 1 to estimate the elements of the adjacency matrix $A = (a_{ij})$. We conclude from Eq. (6) that

$$x_{is} - \hat{x}_{is} = \psi_i/\theta_i, \forall i \in V \tag{8}$$

where

$$\psi_i := \sum_{j\in V, j\neq i} a_{ij}[h_j(x_{js}) - h_i(x_{is})] + f_i(x_{is}),$$

which in combination with Eq. (7) and the Lipschitzian properties of f_i and h_i implies that

$$|\psi_i| \le \left[L_{1i} + \sum_{j\in V, j\neq i} a_{ij}(L_{2j} + L_{2i})\right] \sqrt{\frac{2\theta_{max}}{\theta_{min}} \sum_{k\in V} \hat{x}_{ks}^2}. \tag{9}$$

On the other hand, Eq. (6) also indicates that for all $i \in V$,

$$\sum_{j\in V, j\neq i} a_{ij}[h_j(\hat{x}_{js}) - h_i(\hat{x}_{is})] = \theta_i(x_{is} - \hat{x}_{is}) - f_i(\hat{x}_{is}) + \Delta_i, \tag{10}$$

wherein

$$\Delta_i := \sum_{j\in V, j\neq i} a_{ij}[h_j(\hat{x}_{js}) - h_j(x_{js}) - h_i(\hat{x}_{is}) + h_i(x_{is})] + f_i(\hat{x}_{is}) - f_i(x_{is}), \tag{11}$$

satisfying

$$\begin{aligned} |\Delta_i| &\le \sum_{j\in V, j\neq i} a_{ij}(L_{2j}|\hat{x}_{js} - x_{js}| + L_{2i}|\hat{x}_{is} - x_{is}|) + L_{1i}|\hat{x}_{is} - x_{is}| \\ &\le \sum_{j\in V, j\neq i} a_{ij}(L_{2j}|\psi_j|/\theta_j + L_{2i}|\psi_i|/\theta_i) + L_{1i}|\psi_i/\theta_i| \\ &\le \frac{\left[L_{1i} + \sum_{j\in V, j\neq i} a_{ij}(L_{2j} + L_{2i})\right] \cdot \max_{k\in V}|\psi_k|}{\theta_{min}} \\ &\le \frac{\Omega_{max}^2}{\theta_{min}} \sqrt{\frac{2\theta_{max}}{\theta_{min}} \sum_{k\in V} \hat{x}_{ks}^2} = \frac{\Omega_{max}^2 \sqrt{\sum_i \hat{x}_{is}^2}}{\sqrt{\theta_{min}^3/(2\theta_{max})}} \end{aligned} \tag{12}$$

where the first step has used the Lipschitzian properties of f_i and h_i, and

$$\Omega_{max} := \max_{i\in V}\left\{\left[L_{1i} + \sum_{j\in V, j\neq i} a_{ij}(L_{2j} + L_{2i})\right]\right\}. \tag{13}$$

Equation (12) results in $|\Delta_i| \approx 0$ when large enough gains θ_i are used such that

$$\theta_{min}^3/(2\theta_{max}) \gg \Omega_{max}^4 \sum_i \hat{x}_{is}^2 \tag{14}$$

where Ω_{max} is given by (13).

In this case, Eq. (10) actually leads to

$$\sum_{j\in V, j\neq i} a_{ij}[h_j(\hat{x}_{js}) - h_i(\hat{x}_{is})] \approx \theta_i(x_{is} - \hat{x}_{is}) - f_i(\hat{x}_{is}), \forall i. \tag{15}$$

For any j, we set $\hat{x}_{ks}$ as:

$$\hat{x}_{ks} = \begin{cases} \delta_j, & \text{for } k = j, \\ 0, & \text{otherwise} \end{cases} \tag{16}$$

which in combination with $h_i(0) = 0$ implies that for all $i \in V\backslash\{j\}$,

$$h_k(\hat{x}_{ks}) - h_i(\hat{x}_{is}) = \begin{cases} \beta, & \text{for } k = j, \\ 0, & \text{otherwise} \end{cases} \tag{17}$$

where δ_j is chosen such that $\beta := h_j(\delta_j) \neq 0$.

Substituting Eqs. (16) and (17) into Eq. (15) leads to

$$a_{ij}\beta \approx \theta_i x_{is}, \qquad \forall i \in V\backslash\{j\} \tag{18}$$

where $f_i(0) = 0$ for all i are used. This indicates

$$a_{ij} \approx \theta_i x_{is}/\beta \tag{19}$$

for any $i \in V\backslash\{j\}$. It follows that when conditions (5) and (14) are satisfied, the *driving* connections of any node j can be estimated by the following equation

$$\hat{a}_{ij} = \Xi\left[\frac{\theta_i x_{is}}{\beta}\right], \qquad \forall i \in V\backslash\{j\} \tag{20}$$

where the operation $\Xi(y) = \begin{cases} 1, & \text{for } y \ge 0.5 \\ 0, & \text{for } y < 0.5 \end{cases}$ is used for rounding the element y to the nearest integers in the set $\{0, 1\}$ because the right-hand side (RHS) of (19) in general is a real number. Hence, either 0 or 1 is finally fixed for $\hat{a}_{ij}$.

That $\beta = h_j(\delta_j)$ in Eq. (20) is known *a priori* is really a restriction in practice. We now show that this drawback can be removed by using a β-estimator that approximates β with high accuracy. Indeed, if node j is not isolated, then there exists $k \in V\backslash\{j\}$ such that $a_{kj} = 1$. In this case, Eq. (18) indicates that β can be estimated by the following equation

$$\hat{\beta} = \theta_k x_{ks}, \tag{21}$$

where $\hat{\beta}$ is an estimation of true β (in practice one calculates $\theta_w x_{ws}$ for all $w \in V\backslash\{j\}$ and chooses any w as k and $\hat{\beta} = \theta_w x_{ws}$ if $\theta_w x_{ws}$ does not approximate to zero). This in combination with Eq. (20) implies

$$\hat{a}_{ij} = \Xi\left[\frac{\theta_i x_{is}}{\hat{\beta}}\right], \quad \forall i \in V\backslash\{j\} \tag{22}$$

which indicates that an improved driving connection estimation method has been suggested.

2.3. Estimating node dynamics

We now show that the steady state response equation set (6) can be extended to estimate node dynamics of *balanced*[1] networks. It is easy to verify that

$$\begin{aligned}
&\sum_{i\in V}\sum_{j\in V, j\neq i} a_{ij}[h_j(x_{js}) - h_i(x_{is})] \\
= &\sum_{i\in V}\sum_{j\in V, j\neq i} a_{ij}h_j(x_{js}) - \sum_{i\in V} h_i(x_{is})\Bigg(\sum_{j\in V, j\neq i} a_{ij}\Bigg) \\
= &\sum_{j\in V} h_j(x_{js})\Bigg(\sum_{i\in V, i\neq j} a_{ij}\Bigg) - \sum_{j\in V} h_j(x_{js})\Bigg(\sum_{i\in V, i\neq j} a_{ji}\Bigg) \\
= &\sum_{j\in V} h_j(x_{js})\Bigg(\sum_{i\in V, i\neq j} a_{ij} - \sum_{i\in V, i\neq j} a_{ji}\Bigg) = 0, \qquad (23)
\end{aligned}$$

where the last step has used the property of *balanced networks*. This indicates that summing Eq. (6) over i yields

$$\begin{aligned}
0 &= \sum_{i\in V}[\theta_i(x_{is} - \hat{x}_{is}) - f_i(x_{is})] \\
&= \sum_{i\in V}[\theta_i(x_{is} - \hat{x}_{is}) - f_i(x_{is}) + f_i(\hat{x}_{is}) - f_i(\hat{x}_{is})].
\end{aligned}$$

It follows

$$\sum_{i\in V} f_i(\hat{x}_{is}) = \sum_{i\in V}[\theta_i(x_{is} - \hat{x}_{is}) - f_i(x_{is}) + f_i(\hat{x}_{is})].$$

When the Lipschitzian property of f_i and $\theta_i \gg L_{1i}$ are used, this implies

$$\sum_{i\in V} f_i(\hat{x}_{is}) \approx \sum_{i\in V} \theta_i(x_{is} - \hat{x}_{is}). \qquad (24)$$

For *any* given i, we set $\hat{x}_{ks} = 0$ for $k \neq i$ such that in the summation in the left-hand side (LHS) of Eq. (24) there remains only the item $f_i(\hat{x}_{is})$ and thus obtain

$$\hat{f}_i(\hat{x}_{is}) = \theta_i(x_{is} - \hat{x}_{is}) + \sum_{k\in V, k\neq i} \theta_k x_{ks}, \qquad (25)$$

where $\hat{f}_i$ is an estimation of f_i. Using fitting methods, we can thereby achieve $\hat{f}_i$ to approximate f_i with any precision when $\hat{x}_{is}$ is gradually changed with a small enough rate in a desired range.

2.4. Estimating coupling functions

Equation (18) actually reads

$$a_{ij}h_j(\delta_j) \approx \theta_i x_{is}, \quad \forall i \in V\backslash\{j\}. \qquad (26)$$

If the node j is not isolated, then there exists $k \in V\backslash\{j\}$ such that $a_{kj} = 1$. In this case, Eq. (26) indicates

$$\hat{h}_j(\delta_j) \approx \theta_k x_{ks}, \qquad (27)$$

[1]For all i, $\sum_{k\in V, k\neq i} a_{ik} = \sum_{k\in V, k\neq i} a_{ki}$. All symmetric networks are *balanced*.

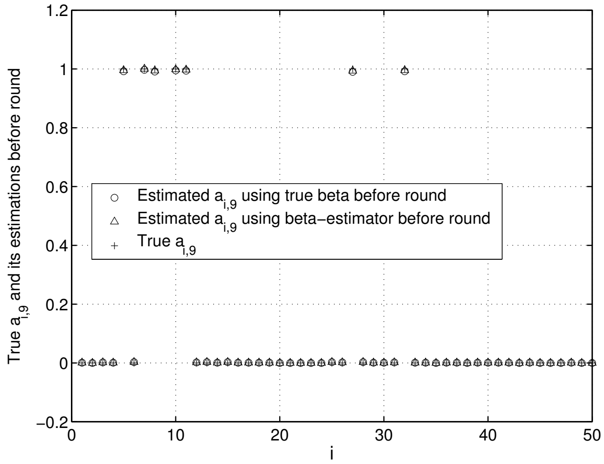

Figure 1: True $a_{i,9}$ (+) and its estimations using true β (○) as well as using β-estimator (Δ) before rounding for all $i \neq 9$.

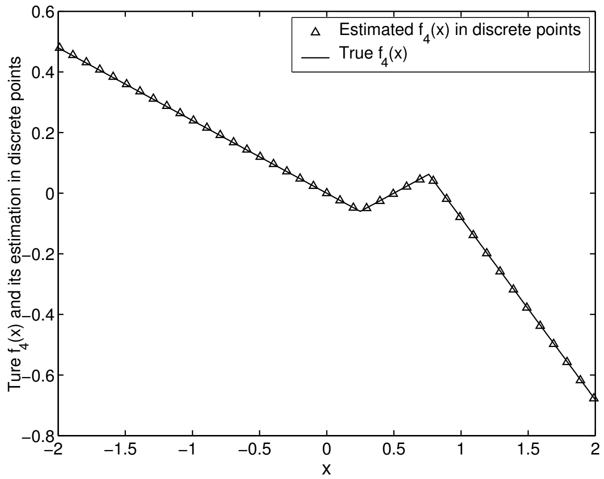

Figure 2: True $f_4(x)$ (—) for $x \in [-2, 2]$ and its approximation $\hat{f}_4(x)$ (△) in discrete points.

where $\hat{h}_j$ is an estimation of h_j (in practice one first calculates $\theta_w x_{ws}$ for all $w \in V\backslash\{j\}$ and then chooses any w as k and $\hat{h}_j(\delta_j) = \theta_w x_{ws}$ if $\theta_w x_{ws}$ does not approximate to zero). By using fitting methods, we can thereby approximate h_j with any precision when δ_j is gradually changed with a small enough rate in a desired range.

Remark: For sparsely connected networks, the conditions (5) and (14) can easily be fulfilled because most elements a_{ij} are zero. This implies that the topology detection method works very well for regular networks, small networks [6], local coupling networks and modular networks. Following similar steps as for the 1D case, one can identify connection topology, individual node dynamics, and coupling functions of dynamical networks with high-dimensional elements.

3. Examples

To illustrate our method, we consider a Cellular Neural Network (CNN) of 1D cells [7], where $h_i(x_i) = (1 - e^{-2\eta_i x_i})/(1 + e^{-2\eta_i x_i})$ and

$$f_i(x_i) := \begin{cases} -\nu_{1i}x_i, & \text{for } x_i < \alpha_{1i}, \\ \nu_{1i}x_i - 2\nu_{1i}\alpha_{1i}, & \text{for } \alpha_{1i} \le x_i \le \alpha_{2i}, \\ -\nu_{2i}x_i - 2\nu_{1i}\alpha_{1i} + (\nu_{1i} + \nu_{2i})\alpha_{2i}, & \text{for } x_i > \alpha_{2i}. \end{cases}$$

In the following, we assume that η_i, α_{1i}, α_{2i}, ν_{1i}, ν_{2i} and τ_{ij} are uniformly distributed in $[0.5, 2]$, $[0.1, 0.3]$, $[0.7, 0.9]$, $[0.2, 0.4]$, $[0.4, 0.8]$, and $[0, 1]$, respectively. We show the typical results for small-world network model [6], which is constructed by the following rules: (I) Start with a m-nearest-neighbor coupled regular network and assume $1 < m << n$; and (II) Add a new long-range edge into the network with probability $0 < p << 1$ between randomly chosen pairs of nodes. We set $n = 50$, $m = 4$, and $p = 0.1$.

Setting $\hat{x}_{ks}$ chosen as in (16) with $\delta_j = 2$, we can determine the driving connections of any node j from Eq. (20) when $\beta = h_j(\delta_j)$ is known, or from Eq. (22) when $h_j(\delta_j)$ is unknown and β-estimator (21) is used. Figure 1 shows the comparative results of the driving connections estimated for node $j = 9$ as a representative node by using true β and β-estimator, respectively. The true values for $a_{i,9} (i \neq 9)$ are: $a_{i,9} = 1$ when $i = 5, 7, 8, 10, 11, 27, 32$, and $a_{i,9} = 0$ otherwise. Then it is easy to see from Fig. 1 that both Eq. (20) and Eq. (22) can identify correctly the driving connections of node $j = 9$.

We can estimate any node dynamics f_i from Eq. (25) when $\hat{x}_{is}$ is gradually changed with a small enough step rate in a desired range. As a typical result Fig. 2 shows the node dynamics estimation for the node $i = 4$ when $\hat{x}_{is}$ starts from -1 and increases with rate 0.1 per step. Similarly one can achieve estimated node dynamics in any given range [a, b] with high accuracy.

Finally we show the coupling function estimation using Eq. (27) wherein δ_j is gradually changed with a small enough step rate in a desired range. As a representative example Fig. 3 shows the coupling function estimation for the node $j = 3$ when δ_j starts from -2 and increases with rate 0.1 per step. One can similarly obtain estimated output functions in any given range [a, b] with high accuracy.

4. Conclusions

We suggested a pinning control based method for identifying the architecture of dynamical networks with unknown time-varying interaction delays. This architecture identification method can be applied to: i) better understand and predict the cooperative dynamic behavior and emerging functions of dynamical networks; ii) detect network "fault" caused by the network architecture; and iii) perturb the connection topology, coupling functions, or node dynamics for regulating the cooperative dynamic behavior and functions of networks as desired.

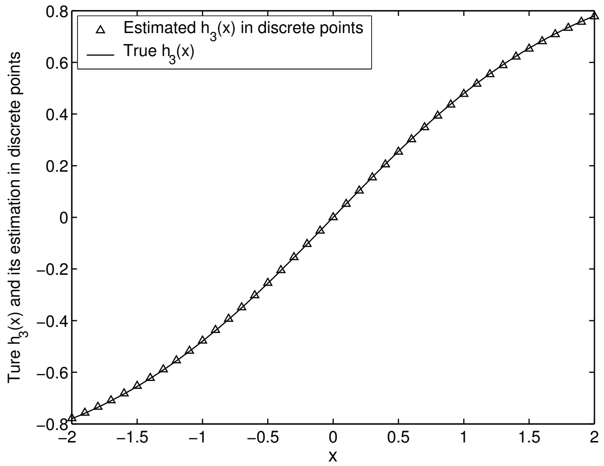

Figure 3: True $h_3(x)$ (—) for $x \in [-2, 2]$ and its approximation function $\hat{h}_3(x)$ (△) in discrete points.

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