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Appearance of the Grazing Bifurcation in an Impact Model with Three Mass Points

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Abstract—In this paper, we analyze an impact model with three masses numerically and experimentally. It replicates the dynamics of the gear system. First, we explain the model and its dynamics and then introduce the experimental setup. Finally, we show some numerical and experimental results. In particular, we find the appearance of the grazing bifurcation in this model.

1. Introduction

Dynamical system with interrupted characteristics has been analyzed since decades [1]. It is well known that this class of the systems exhibit various kinds of the bifurcation phenomena, e.g., period doubling bifurcation [2, 3], border-collision bifurcation [4, 5], and so on. In particular, it is considered that the bifurcation analysis is a effective way to clarify the system's qualitative property and many researchers have analyzed it in detail.

Impact oscillators are often observed in the engineering field. The impact damper or over head wire-pantograph system in the mechanical field [6, 7], spiking neuron model in the biological field [8] and forest fire model in the ecological field [9] are the typical examples of the practical impact oscillators. On the other hand, it is also known that most of the mechanical systems have some gap between each apparatus in order to smoothly put themselves into action; the gear system is the typical example of it. In general, this class of the mechanical systems have the unavoidable problem, i.e., impact oscillation occurs when each apparatus run with the compelling force. Moreover, above problem affect us seriously in our daily lives, such as the noise, attrition, and so on. Thus, to examine the fundamental mechanism of the impact oscillation in the gear system is one of the important topic from the practical point of view [10, 11]. But, the detailed analysis of the practical gear system is very difficult because of its complex behavior. Therefore, Ref. [12] proposed the simplest class of the impact model which simulates the dynamics of the practical gear systems by using three mass points, springs and dampers. The basic oscillation has been discussed in Ref. [12]; however, detailed analysis and its experimental verification is insufficient.

This paper addresses the first step to examine the bifurcation phenomena in an impact model with three mass points, both numerically and experimentally. First, we explain the model and its dynamics and then introduce the experimental setup. Finally, we show some numerical and experimental results. In particular, we find the grazing bifurcation in this model.

2. Impact models with three mass points

2.1. A physics model

Figure 1 shows a physical model of the gear system [10, 11]. The model has three mass points, springs and dampers; m_1, m_2 and m_3 denote the mass, k_1, k_2 and k_3 denote the spring constant, c_1, c_2 and c_3 denote the damping coefficient, respectively. e_1 and e_2 are the gap between the mass points. The compelling force $F = a\cos\omega t$ is impressed on the mass point m_2, and impact oscillation occurs between m_2 and m_1 or m_2 and m_3.

The motion equations are defined as follows:

$$\begin{cases} m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 = 0 \\ m_2\ddot{x}_2 + c_2\dot{x}_2 + k_2x_2 = acos(\omega t) \\ m_3\ddot{x}_3 + c_3\dot{x}_3 + k_3x_3 = 0 \end{cases}, \quad (1)$$

where x_1, x_2 and x_3 is the displacement from the equilib-

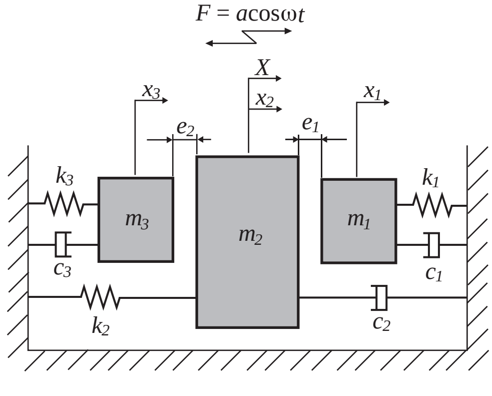


Figure 1: A gear system.

rium position. Now, we define the following equation in order to unify the displacement coordinate with m_2.

$$X_1 = x_1 + e_1, \quad X_2 = x_2, \quad X_3 = x_3 - e_2 \tag{2}$$

The dimensionless values are set as follows:

$$\begin{aligned} C_1 &= \frac{c_1}{m_1},\ C_2 = \frac{c_2}{m_2},\ C_3 = \frac{x_3}{m_3}, \\ K_1 &= \frac{k_1}{m_1},\ K_2 = \frac{k_2}{m_2},\ K_3 = \frac{k_3}{m_3},\ A = \frac{a}{m_2}. \end{aligned} \tag{3}$$

Thus, Eq. (1) is rewritten as follows:

$$\begin{cases} \ddot{X}_1 + C_1\dot{X}_1 - K_1X_1 - K_1e_1 = 0 \\ \ddot{X}_2 + C_2\dot{X}_2 + K_2X_2 = Acos(\omega t) \\ \ddot{X}_3 + C_3\dot{X}_3 + K_3X_3 + K_3e_2 = 0 \end{cases}. \tag{4}$$

In general, the impact phenomenon is classified into two types in the model; the first one is the elastic collision, and the other is the inelastic collision. Note that the model which is analyzed in this paper exhibits inelastic collision. Thus, the impact phenomenon is calculated by the relationship between the law of conservation of momentum and the reflection coefficient, i.e., the following equations are defined.

$$\begin{cases} m_1\dot{X}_{1+} + m_2\dot{X}_{2+} = m_1\dot{X}_{1-} + m_2\dot{X}_{2-} \\ \dot{X}_{1+} - \dot{X}_{2+} = -E(\dot{X}_{1-} - \dot{X}_{2-}) \end{cases}, \tag{5}$$

$$\begin{cases} m_3\dot{X}_{3+} + m_2\dot{X}_{2+} = m_3\dot{X}_{3-} + m_2\dot{X}_{2-} \\ \dot{X}_{3+} - \dot{X}_{2+} = -E(\dot{X}_{3-} - \dot{X}_{2-}) \end{cases} \tag{6}$$

$\dot{X}_{1-}$, $\dot{X}_{2-}$ and $\dot{X}_{3-}$ are the velocities of m_1, m_2 and m_3 before the impact. $\dot{X}_{1+}$, $\dot{X}_{2+}$ and $\dot{X}_{3+}$ are the velocities of m_1, m_2 m_3 after the impact. E in Eqs. (5) and (6) denote the reflection coefficient between the each mass points. Consequently, Eqs. (5) and (6) can be written as follows.

$$\begin{cases} \dot{X}_{1+} = \dfrac{m_1 - Em_2}{m_1 + m_2}\dot{X}_{1-} + \dfrac{(1+E)m_2}{m_1 + m_2}\dot{X}_{2-} \\ \dot{X}_{2+} = \dfrac{(1+E)m_1}{m_1 + m_2}\dot{X}_{1-} + \dfrac{m_2 - Em_1}{m_1 + m_2}\dot{X}_{2-} \end{cases} \tag{7}$$

$$\begin{cases} \dot{X}_{3+} = \dfrac{m_3 - Em_2}{m_3 + m_2}\dot{X}_{3-} + \dfrac{(1+E)m_2}{m_3 + m_2}\dot{X}_{2-} \\ \dot{X}_{2+} = \dfrac{(1+E)m_3}{m_3 + m_2}\dot{X}_{3-} + \dfrac{m_2 - Em_3}{m_3 + m_2}\dot{X}_{2-} \end{cases} \tag{8}$$

Eqs. (7) and (8) denote the velocity of the mass points m_1, m_2 and m_3 after the impact.

Figure 2 shows an example of the orbit. In the figure, x_{m1}, x_{m2} and x_{m3} denote the displacement, v_{m1}, v_{m2} and v_{m3} denote the velocity of the mass points m_1, m_2 and m_3, respectively. The dynamics of the orbits is given by Eq. (4). When the orbit x_{m2} hits x_{m1}'s orbit, the velocity of the mass point v switches from v_- to v_+. After that, x_{m2} exhibits free oscillation, x_{m1} behaves with the compelling force. Note that the relationship between x_{m2} and x_{m1} is same.

2.2. Experimental setup

To examine the dynamics of the gear system experimentally, we propose an experimental apparatus which simulates the dynamics of the gear system. Figure 3 shows the conceptual diagram of the model. Mass1, Mass2 and Mass3 denote the mass points m_1, m_2 and m_3 in Fig. 1, respectively. The spring constants are set as k_1, k_2 and k_3. The gaps between Mass1 and Mass2, e_1, and Mass2 and Mass3, e_2, are set as $e_1 = 2$[mm] and $e_2 = 1$[mm]. The motor (TAMIYA 6-Speed Gearbox H. E.), which is fixed on Mass2, gives the compelling force to Mass2; the compelling force is described as $F = a\cos\omega t$. Note that each mass oscillate with single degree of freedom on the rail. In addition, Mass1, Mass2 and Mass3 are made of different material in each impact plane; Mass1 and Mass3 are made of aluminum block (MISUMI A5052P) and Mass2 is made of the steel (MISUMI SS400). The displacement of each mass is determined by the laser displacement sensor (KEYENCE LK-G155 LK-G405).

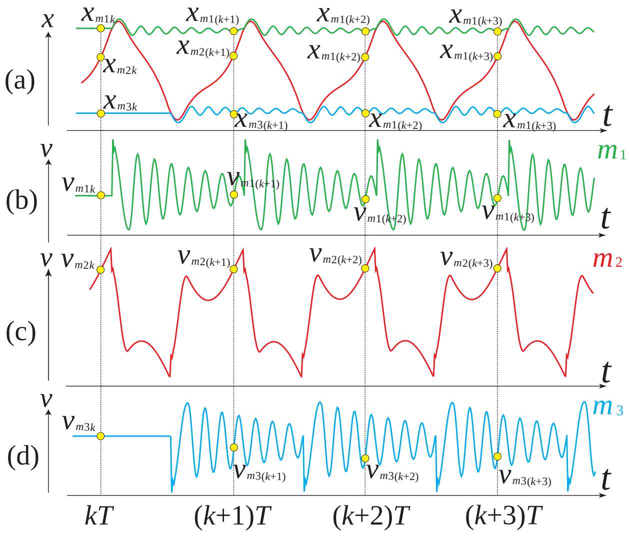


Figure 2: An example of the orbit.

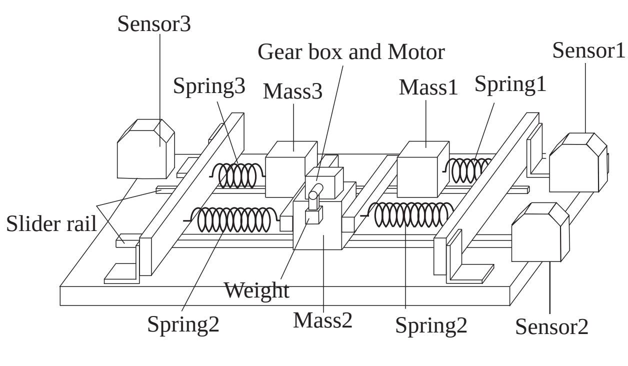
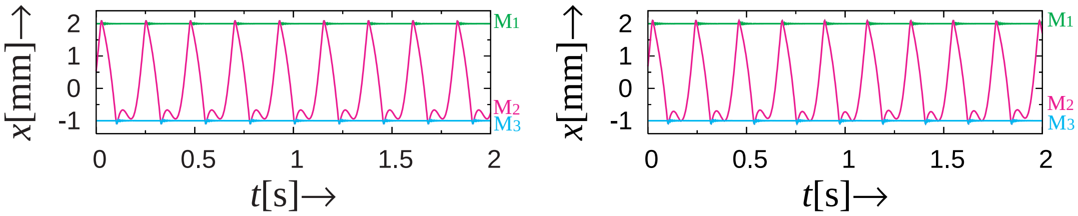
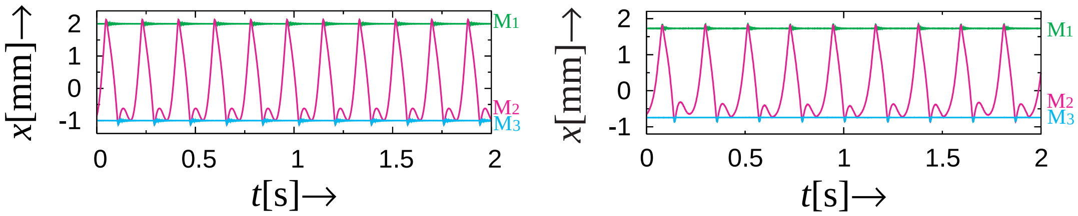


Figure 3: An impact model with three mass points.



(a) Periodic waveform (f = 4.4 [Hz]) (b) Non-periodic waveform (f = 4.6 [Hz])

Figure 4: An example of the orbit in the numerical result.



(a) Periodic waveform (f = 4.4 [Hz]) (b) Non-periodic waveform (f = 4.6 [Hz])

Figure 5: An example of the orbit in the experimental result.

3. Analytical results

3.1. Dynamical behavior of the experimental apparatus

Figures 4 and 5 show an example of the orbit; Fig. 4 is the numerical result, Fig. 5 is the experimental result. In these figures, the green, red and blue-colored orbits denote the time series displacement of Mass1, Mass2 and Mass3. Note that we assume the upwards displacement as the positive direction. It is clear that the numerical results are verified by the experimental result; because the experimental result duplicates the system dynamics as the numerical result at the same parameter value. Figure 6 shows an example of the one parameter bifurcation diagram with changing the frequency of the compelling force f. Note that the orbit is mapped by every period of T which is the cycle of the compelling force. It should be also mentioned that (a) and (b) in Figs. 4 and 5 correspond to the parameters (a) and (b) in Fig. 6. It can be understood that the model has various kinds of periodic orbit and non-periodic orbit. For example, the periodic orbit is observed around f = 4.5[Hz] (see Figs. 4, 5 and 6 (a)). After that the periodic orbit bifurcates to the non-periodic orbit as the parameter f is increased (see Figs. 4, 5 and 6 (b)). The same system behavior appears at the other parameter value of f as shown in Fig. 6. Thus, it is said that the bifurcation has occurred and the system dynamics has changed with change in the compelling force f. In the following analysis, we focus on the bifurcation phenomena and its effect on the system behavior.

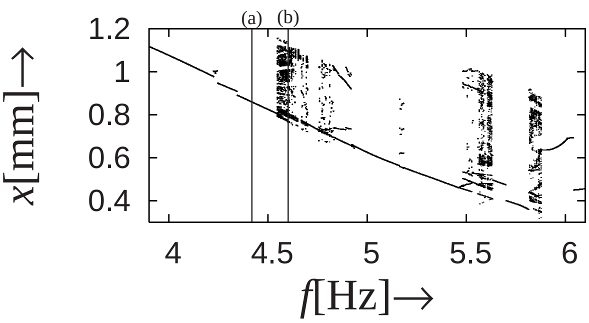


Figure 6: 1-dimensional bifurcation diagram

3.2. Appearance of the grazing bifurcation

We focus on Figs. 4 and 5 (a). It is clear that Mass2 exhibits periodic behavior in free oscillation with alternate impact phenomenon between Mass1 and Mass2 in both of the numerical and experimental results. On the other hand, the non-periodic orbit is observed at f = 4.6[Hz] (see Figs. 4 and 5 (b)). There are two types of the different behavior after the first impact phenomenon between Mass2 and Mass3 at f = 4.6[Hz]; Mass2 hits Mass1 again in the first case, Mass2 hits Mass3 in the other case. Thus, Figs. 4 and 5 (b) seems to be the periodic orbit; however, it is classified into the non-periodic orbit because the impact between Mass2 and Mass3 has no regularity. This kind of system behavior is seen in the other parameter value of f (see Fig. 6). We find that the grazing bifurcation affects the system dynamics very much. To examine the mechanism of the grazing bifurcation in detail, we show a conceptual diagram of the route to the grazing bifurcation in Fig. 7. The upper side of the orbit denote Figs. 4 and 5 (a) and the lower side of the orbit denote Figs. 4 and 5 (b), respectively. It is clear that the impact oscillation has occurred and the system dynamics has changed. In other words, it

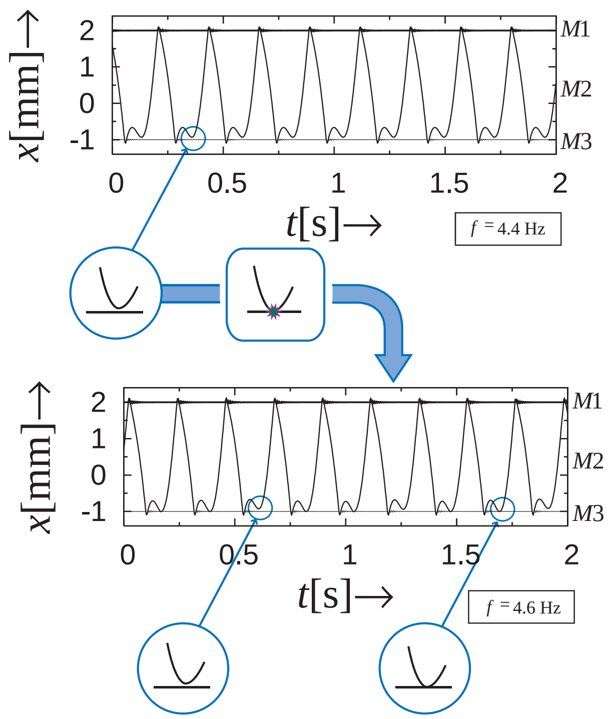


Figure 7: Appearance of the grazing bifurcation.

is understood that the grazing bifurcation is occurred between the parameter value of (a) and (b) in Figs. 4, 5 and 6. We predict that all of the bifurcation phenomena which can be observed in the system is the grazing bifurcation. Consequently, we conclude that the appearance of the grazing bifurcation greatly affect in the system dynamics.

4. Conclusion

In this paper, we have analyzed an impact model with three masses, which simulates the dynamics of the gear system both from the numerical and experimental points of view. First, we explained the basic model of the gear system and its dynamics and then introduced the experimental setup. Finally, some numerical and experimental results were shown. We found that the system has various kinds of the periodic orbit and non-periodic orbit. In particular, it was clarified that the appearance of the grazing bifurcation greatly affected in the system dynamics. The analytical results will contribute to the practical gear system; because the model simulated the basic dynamics of the practical gear systems.

Further, we plan to examine the system's qualitative property in a wide parameter space via the bifurcation analysis.

Acknowledgments

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