2010 International Symposium on Nonlinear Theory and its Applications
NOLTA2010, Krakow, Poland, September 5-8, 2010

Synchronization regulation of a complex network by link rewiring or node pinning

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Abstract—Synchronization is essential for the correct functioning of many technological systems as in wireless communications, parallel/distributed computing, or electrical power distribution lines. One of the properties that those systems should have is scalability, i.e., the ability to be augmented without loosing the synchronization performance. In this communication, we describe a method to properly engineer the wiring of a network of dynamical units in order to achieve synchronization. We provide rules to guide the rewiring of the links of a given node in the network or to guide the pinning with a new only node added to the network. In particular, we will focus on sequential regulation processes in order to establish conditions for identifying the minimal number of connections needed for regulating synchrony, as well as a practical way to find the corresponding sequence of connections.

1. Introduction

Regulation of synchronization of networking dynamical units is an issue of the outmost importance because synchronous behavior is a must for the correct functioning of many technological and biological networks [1]. As concerning the former class of networks, synchronous message passing in computer science, is a form of communication used in parallel/distributed computing [2] or in wireless sensor networks [3], while in the later class, a quorum-sensing mechanism is used in many populations of cells to cause expression of genes in gene regulatory networks [4, 5].

A common feature of these systems is that they are continuously growing. Thus, it is essential to understand how this growth it is accompanied without loosing the synchronization performance, in order to optimally design technological networks or synthetic biological gene networks.

Here we propose pinning regulation as a model to understand the regulatory mechanisms underlying this synchronous behavior of a network of dynamical units [6]. We provide a full description on how to engineer an external pinning action on a network of identical dynamical units leaving unchanged its dynamical properties and topology and minimally acting to achieve regulation of synchrony. The pinning interaction comes from an external node which is identical to the nodes of the network and whose only effect is establishing bidirectional connections of the same strength in sequential steps. Our aim is to provide the conditions to achieve this with the minimal number of connections.

2. Pinning regulation

In order to evaluate the pinning regulation of a given network, we consider an initial graph $\mathcal{G}_0$ of N bidirectional coupled identical systems each one represented by a m-dimensional real vector state $\mathbf{x}_i$ ($i = 1, \ldots, N$), whose evolution is given by the equation:

$$\dot{\mathbf{x}}_i = f(\mathbf{x}_i) + \sigma \textstyle\sum_{j=1}^{N} \mathcal{L}_{ij} h(\mathbf{x}_j),$$

and depends on the local function f, on the coupling function h, on the Laplacian matrix $\mathcal{L} \in \mathcal{M}_N$ associated to the connectivity described by the graph $\mathcal{G}_0$, and on the fixed coupling strength σ. The assumption of a network made of identical systems and a zero-row sum Laplacian ensures the existence of a synchronous state $[\mathbf{x}_1(t) = \mathbf{x}_2(t) = ... = \mathbf{x}_N(t) = \mathbf{x}_s(t)]$ whose stability can be studied by means of the Master Stability Function (MSF) approach [7]. The MSF approach demonstrates that there are only two classes of systems for a given local and coupling functions that allow stability of the synchronous state [8].

Figure 1 shows the shapes of the maximum Lyapunov exponent Λ for the so called class II, Fig. 1 (central column), and class III systems, Fig. 1 (right column), as a function of a parameter λ which essentially depends on f and h [8]. For class II systems, the synchronous state $\mathbf{x}_s$ is stable above a critical $\lambda_c^1 = \frac{\mu_1}{\sigma}$ as Λ is a monotonically decreasing function, while stability for class III with a V-shape Λ function is reached within an interval of values $\lambda_c^1 = \frac{\mu_1}{\sigma}$ and $\lambda_c^2 = \frac{\mu_2}{\sigma}$ In our case, as the coupling strength σ is fixed, the variable moving along the x-axis is directly related to the eigenvalues of the Laplacian matrix.

We start from a situation in which the network topology gives rise to an unstable synchronous state (as depicted at

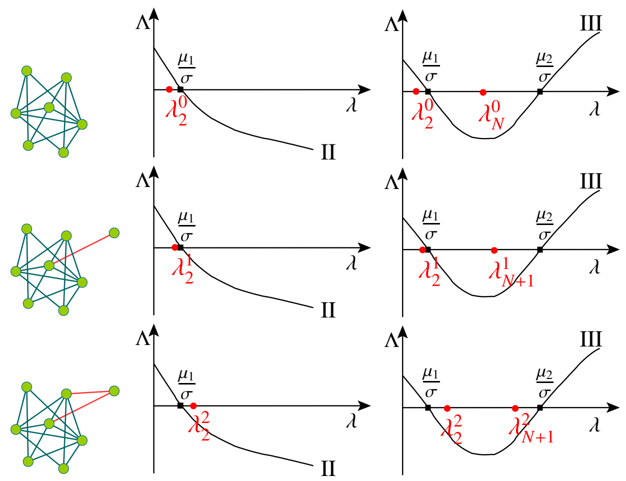

Figure 1: Possible classes of master stability function for networked chaotic systems. In all cases $(\Lambda(\lambda = 0) > 0)$ is the maximum Lyapunov exponent of the single uncoupled system. The case II (middle column) corresponds to a monotonically decreasing master stability function. Case III (right column) admits a finite range of negative values for $\Lambda(\lambda)$. The network (left column) capability to give rise to a stable synchronized dynamics depends on the distribution of its eigenvalues spectrum along the λ axis: $\lambda_2 > \frac{\mu_1}{\sigma}$ for class II and both λ_2 and λ_{N+1} within the interval $(\frac{\mu_1}{\sigma}, \frac{\mu_2}{\sigma})$.

the top row of Fig. 1), that is, if we order the eigenvalues (which is possible because the Laplacian matrix is zero-row sum and symmetric), $0 = \lambda_1^0 < \lambda_2^0 < \ldots < \lambda_N^0$, the smallest non zero eigenvalue, λ_2^0, is outside the stability region for both classes of systems. For the class III system besides, the largest eigenvalue, λ_N^0, has to be initially located inside the stability region in order to be able to regulate the network to synchrony. Otherwise, if $\lambda_N^0 > \frac{\mu_2}{\sigma}$, the synchronous state is impossible to turn stable by pinning.

In order to regulate the stability of $\mathbf{x}_s(t)$, we consider here an interaction between $\mathcal{G}_0$ and an external dynamical system, identical to those in $\mathcal{G}_0$, that forms, at successive times $t_n\,(n = 1, ..., N)$ a series of σ-strength connections by pinning the nodes in $\mathcal{G}_0$ with a given sequence $\{s_1, s_2, \ldots, s_N\}$. This is schematically shown in Fig. 1 at the left column for t_1 (middle row) and t_2 (bottom row), where the original graph is being pinned by an external node (red links).

This situation is now described by this new equation of motion,

$$\dot{\mathbf{x}}_i = f(\mathbf{x}_i) + \sigma \sum_{j=1}^{N+1} \mathcal{L}'_{ij}(t) h(\mathbf{x}_j), \qquad (1)$$

where $\mathcal{L}'(t) = \left(\mathcal{L}'_{ij}(t)\right) \in \mathcal{M}_{N+1}$ is now the following time dependent Laplacian matrix

$$\left(\begin{array}{cccc|c} \mathcal{L}'_{11}(t) & \mathcal{L}'_{12} & \cdots & \mathcal{L}'_{1N} & \Theta(t-T_1) \\ \mathcal{L}'_{21} & \mathcal{L}'_{22}(t) & \cdots & \mathcal{L}'_{2N} & \Theta(t-T_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{L}'_{N1} & \mathcal{L}'_{N2} & \cdots & \mathcal{L}'_{NN}(t) & \Theta(t-T_N) \\ \hline \Theta(t-T_1) & \cdots & \cdots & \Theta(t-T_N) & \mathcal{L}'_{N+1,N+1}(t) \end{array}\right)$$

whose elements are such that:

i) $\mathcal{L}'_{ij} = \mathcal{L}_{ij}$ for $i \neq j$ and $i, j < N+1$;

ii) $\Theta(t - T_i) = \mathcal{L}'_{i,N+1}(t) = \mathcal{L}'_{N+1,i}(t)$, being T_i the time at which the i^{th} node in $\mathcal{G}_0$ is pinned by the interaction with the external node, and Θ the Heaviside function. Notice that while the index i in t_i refers to a time ordering, the index i in T_i points to the ordering of the pinning sequence, and therefore $t_i = T_{s_i}$.

iii) $\mathcal{L}'_{ii}(t) = -\sum_{j\neq i} \mathcal{L}'_{ij}$.

The effect of pinning the nodes in $\mathcal{G}_0$ with the external one is to produce a new set of eigenvalues for the Laplacian matrix $\mathcal{L}'$, $0 = \lambda_1^n < \lambda_2^n < \ldots < \lambda_N^n < \lambda_{N+1}^n$. It is clear that the only way to regulate the situation depicted at the top row of Fig. 1 by pinning an external dynamical system is moving λ_2 inside the stability region for both class II and class III systems and by keeping the largest eigenvalue (now λ_{N+1}) inside for class III.

While for class II systems, a practical way to regulate the synchronous state is by finding the optimal pinning sequence maximizing λ_2, for the class III the synchronization stability is ensured for all coupling architectures whose corresponding eigenvalue spectrum is entirely contained within the stability region of the MSF, delimited by the two threshold parameters $\frac{\mu_1}{\sigma}$ and $\frac{\mu_2}{\sigma}$. Then, the way we select the pinning sequence is that to maximize (at each time t_n a new link is formed with the regulating node) the distance $\lambda_2^n - \lambda_2^0$, in order to ensure that we are moving to the right in the spectrum and, at the same time, to minimize $\lambda_{N+1}^n - \lambda_N^0$, in order to keep the largest eigenvalue far from the second threshold. This is equivalent to maximize the quantity,

$$R_n = \frac{\lambda_2^n - \lambda_2^0}{\lambda_{N+1}^n - \lambda_N^0}. \qquad (2)$$

Note that, the largest eigenvalue λ_{N+1} always increases by the fact we are adding connections, while is not always true for λ_2.

3. Results

From here on, we will accompany our analytical study using the MSF with numerical examples, corresponding to the case of $N = 400$ nodes arranged in two different network configurations: a small-world network (SW) obtained as in Ref. [9] by initially arranging the N nodes in a ring with connections only between nearest neighbors, and by randomly adding with probability $p = 0.02$

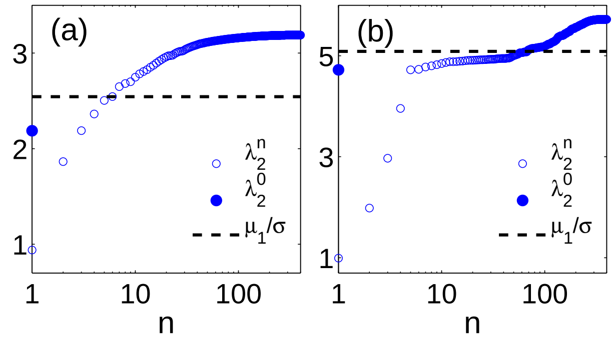

Figure 2: Log-linear graph showing the behavior of λ_2^n during the regulation of a class II system for (a) SW, and (b) SF networks (see text for details). As reference, the values of λ_2^0 and the ratio μ_1/σ with (a) $\sigma = 0.07$ and (b) $\sigma = 0.035$ are given.

a connection between unconnected pairs of nodes (i.e. obtaining an average degree $\langle k\rangle = 2 + pN = 10$), and a scale-free network (SF) obtained by the preferential attachment process of Ref. [10] with the same average degree of $\langle k\rangle = 10$. Furthermore, in all cases, we will consider $f(\mathbf{x} \equiv (x, y, z)) = [-y - z, x + 0.165y, 0.2 + z(x - 10)]$ in Eq. (1) (i.e. we will refer to the case of networks of coupled Rössler systems [11]), because it is known that such a case allows for a direct comparison of class III networks (when $h(\mathbf{x} \equiv (x, y, z)) = [x, 0, 0]$ with $\mu_1 = 0.206$ and $\mu_2 = 5.519$) and class II networks (when $h(\mathbf{x} \equiv (x, y, z)) = [0, y, 0]$ with $\mu_1 = 0.178$).

Let us proceed first with the regulation of the class II system. The behavior of λ_2 by pinning both the SW and the SF networks according to the criterion of maximizing λ_2 is reported in Fig. 2. This figure allows us to identify a minimum number of links between the regulating node and the rest of nodes. In particular, it is sufficient to pin less than 2% of nodes to make the synchronous state stable. At the same time, we observe that, comparing the two complete different networks regarding their degree distribution but with the same average degree, it is evident that the SW is easier to regulate than the SF.

Figure 3 shows the behavior of the smallest (blue open circles) and largest (red open squares) eigenvalues by maximizing the quantity defined by Eq. (2) as a new link is formed between the regulating node and $\mathcal{G}_0$ during the pinning regulation. Note that, before the pinning starts, the initial smallest eigenvalues (blue full face circles) are below the threshold, so the synchronous state is unstable, and the largest ones (red full face squares), are inside the stability region in order to be able to regulate the system. The main difference with respect to the class II system is that the regulation is only possible up to a maximum length sequence determined by the monotonous increase of the largest eigenvalue. Another difference between regulating networks with different heterogeneities in the degree distribution is that the SF allows for a larger number of regulating sequences.

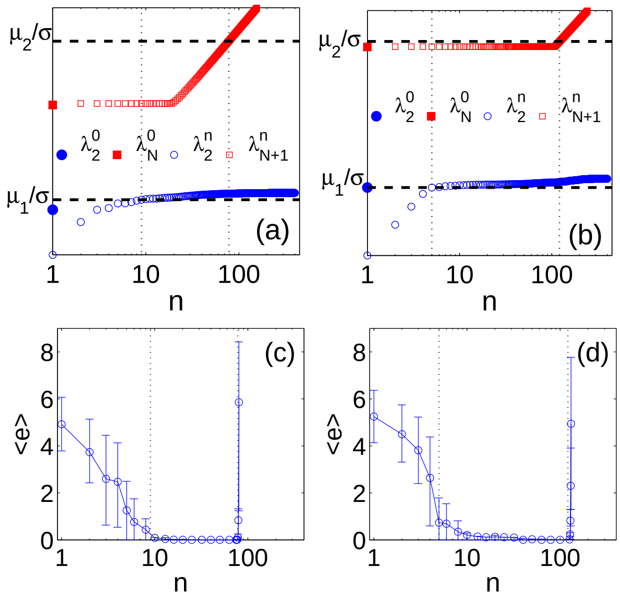

Figure 3: (Color on-line)(a-b) Log-log behavior of λ_2^n (blue open circles) and λ_{N+1}^n (red open squares) during regulation of a class III system for (a) SW and (b) SF networks (same parameters and stipulations as in Fig. 2). The values of λ_2^0 (blue full face circle) and λ_N^0 (red full face square) refer to those of $\mathcal{G}_0$, and the horizontal dashed lines are in correspondence with the limiting thresholds μ_1/σ and μ_2/σ with (a) $\sigma = 0.07$ and (b) $\sigma = 0.035$. (c-d) Log-linear plots of the averaged synchronization errors $\langle e\rangle$ *vs.* n during the regulation of the same (c) SW and (d) SF networks used in (a) and (b) respectively, and the same pinning sequence optimizing R^n given in (a) and (b) respectively. Vertical dashed lines are added to show the agreement with the predicted ranges of regulability depicted in (a-b).

In order to verify the analytical results we performed numerical simulations with a network of coupled chaotic Rössler dynamical units and monitored the time average synchronization error $\langle e\rangle$ with respect to the trajectory of the regulating node. Figure 3(c-d) shows for the SW and SF networks how accurately the numerical results reproduces the vanishing of $\langle e\rangle$ within the range predicted by the theory in Fig. 3(a-b).

To conclude, let us introduce a short discussion regarding our approach of regulation of synchrony. In previous approaches of *pinning control* of networks [13] (i.e. the situation in which the external node is *unidirectionally* forcing the dynamics of the original graph), it was argued that the controllability of a generic network behavior towards an assigned synchronous evolution could be enhanced by pinning configurations that imply a decrease of the ratio $\frac{\lambda_{N+1}}{\lambda_2}$ associated to the extended network topology [14]. This is so because the MSF states that the more packed are the

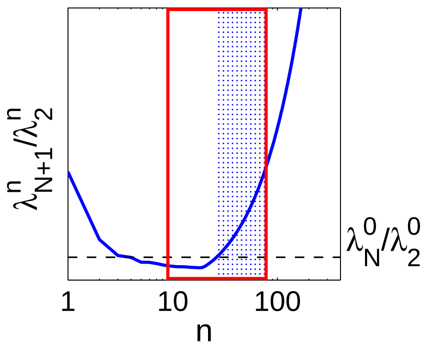

Figure 4: Log-linear behavior of the ratio $\frac{\lambda^n_{N+1}}{\lambda^n_2}$, where λ^n_2 and λ^n_{N+1} are those obtained from the regulation process of the SW network of the class III system reported in Fig. 3(a). The sequences within the red rectangle correspond to all those sequences that stabilize the synchronous state, while those within the shaded region correspond to those that, despite corresponding to values of the eigenratio larger than the initial one (horizontal dashed line) and therefore making less compact the spectra, still correspond to sequences stabilizing the synchronous state.

eigenvalues, that is, the smaller the eigenratio λ_{N+1}/λ_2, the higher the synchronizability of the network. But, if we plot this eigenratio during our pinning regulation process, we observe in Fig. 4 that many pinning sequences exist that are able to regulate the synchronous behavior of $\mathcal{G}_0$ (those within the range of the red rectangle) and that, however, correspond to values of $\frac{\lambda_{N+1}}{\lambda_2}$ that are larger than the initial (unpinned) value, for which the synchronous behavior was unstable (shaded region).

We point out that our conclusions are not limited to processes where regulation is attained by interaction with an external node, but they can also be applied in all cases in which the problem is to regulate synchrony by rewiring the connections of given nodes of a graph. One can, indeed, imagine to start with N networking systems, remove all $\tilde{k}$ connections of a given node, and substitute them with new $\tilde{k}$ connections to the other $N-1$ units of the graph, following the ranking sequence that our criteria are providing, thus enhancing the synchronization behavior of the original graph while maintaining the same number of nodes and links.

Acknowledgments

Work partly supported by EU Contract No. 043309 GABA, by the Spanish Ministry of S&T under Project n. FIS2009-07072, the URJC-CM Project n. 2008-CET-3575, the Community of Madrid under the R&D Program of activities MODELICO-CM/S2009ESP-1691, and by the Chinese NSFC under Grant No. 10602026.

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