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Stability Analysis of Amplitude Death Induced by a Time-Varying Delay Connection in Network Oscillators

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Abstract—The time-delay connection induced amplitude death has been extensively investigated in the field of nonlinear science. Our previous study showed that a time-varying delay connection can induce a stabilization of unstable steady states in a pair of oscillators. This report extends our previous study to network oscillators. A linear stability analysis reveals that such connection is valid even for network oscillators. The analytical results are verified by numerical simulations.

1. Introduction

Various nonlinear phenomena have been observed in coupled nonlinear oscillators, and have been extensively investigated in the field of nonlinear physics [1]. One of the interesting phenomena is amplitude death which is a connection-induced stabilization of an unstable steady state. Amplitude death has been studied theoretically and experimentally for the last quarter-century [2, 3, 4, 5]. It was theoretically confirmed that amplitude death never occurs in diffusively coupled identical oscillators [3, 6]. Reddy *et al.* showed that a transmission delay in connection can induce it [7]. Furthermore, amplitude death has been observed not only in the delay connection but also in conjugate coupling [8], dynamical coupling [9], and nonlinear coupling [10].

Although the time-delay-induced death would be useful for the stabilization of coupled unstable systems, it never occur when the transmission delay is long compared with an oscillation period of the unstable systems. Thus, amplitude death cannot be induced in the following situations: there is a long distance between the oscillators; each oscillator has a high frequency. To overcome the above problems, a distributed delay connection [11] and a multiple delay connection [12] were proposed; however, the former is difficult to realize on electronic circuits and the latter cost twice as much as the conventional time-delay connection. Therefore, a proposal of easy and low cost connections has been expected.

Recently, a time-varying delay connection in which transmission delay varies periodically was proposed [13]. This connection can be implemented by electronic circuits, and takes low cost. As a result, the time-varying connection would be one of the strong candidates for overcoming

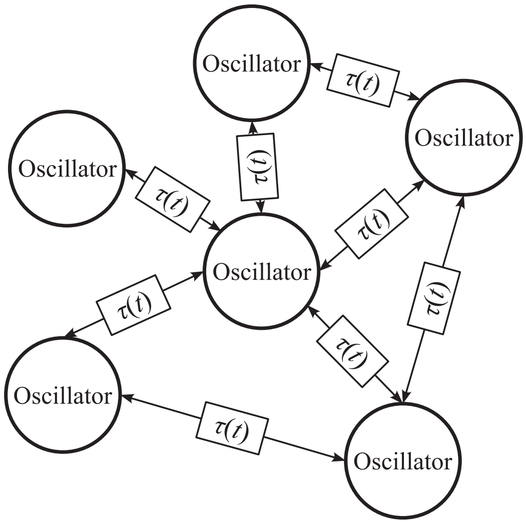


Figure 1: A sketch of network oscillators coupled by the time-varying delay connection

the problem. Unfortunately, the previous study focused on a pair of coupled oscillators; thus, there is a need to extend the previous study to network oscillators (see Fig. 1).

The present paper deals with amplitude death induced by the time-varying delay connection in network oscillators. The stability analysis of amplitude death is investigated for various network topologies. The analytical results are confirmed by numerical simulations.

2. Network oscillators

Let us consider m_x-dimensional oscillators (see Fig. 1),

$$\begin{cases} \dot{\boldsymbol{x}}_i = \boldsymbol{F}(\boldsymbol{x}_i) + \boldsymbol{b}\boldsymbol{u}_i \\ \boldsymbol{y}_i = \boldsymbol{c}\boldsymbol{x}_i \end{cases} \quad (i = 1, 2, \dots, N), \tag{1}$$

where $\boldsymbol{x}_i \in \mathbb{R}^{m_x}$ is the state variable of the ith oscillator and $\boldsymbol{y}_i \in \mathbb{R}^{m_y}$ is the output signal. N is the number of oscillators. $\boldsymbol{b} \in \mathbb{R}^{m_x \times m_u}$ and $\boldsymbol{c} \in \mathbb{R}^{m_y \times m_x}$ are the input and output vectors, respectively. Each oscillator has at least one unstable fixed point $\boldsymbol{x}^* : \boldsymbol{F}(\boldsymbol{x}^*) = 0$. The coupling signal $\boldsymbol{u}_i \in \mathbb{R}^{m_u}$ is

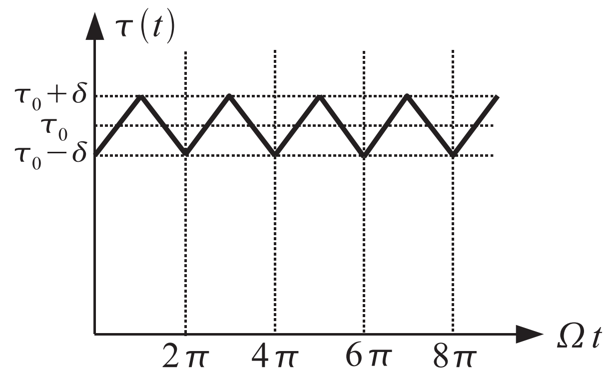


Figure 2: Time-varying delay function $\tau(t)$

described by

$$\boldsymbol{u}_i = \boldsymbol{k}_s \left\{ \frac{1}{d_i} \left(\sum_{l=1}^{N} \varepsilon_{il} \boldsymbol{y}_{l,\tau} \right) - \boldsymbol{y}_i \right\}, \tag{2}$$

where $\boldsymbol{y}_{l,\tau} := \boldsymbol{y}_l(t - \tau(t))$ is the delayed output signal and $\boldsymbol{k}_s \in \mathbb{R}^{m_u \times m_y}$ is the coupling strength. The network topology is governed by ε_{il} as follows: if oscillator i is connected to oscillator l, then $\varepsilon_{il} = \varepsilon_{li} = 1$, otherwise $\varepsilon_{il} = \varepsilon_{li} = 0$. Here $d_i := \sum_{l=1}^{N} \varepsilon_{il}$ denotes the number of oscillators connected to oscillator i. Figure 2 shows the time-varying delay $\tau(t) \geq 0$ which varies periodically around the nominal delay time $\tau_0 > 0$,

$$\tau(t) = \tau_0 + \delta f(\Omega t), \tag{3}$$

where $\delta \in [0, \tau_0)$ and $\Omega > 0$ are the amplitude and frequency of variation, respectively. $f(x)$ is the periodic triangle function whose period is 2π. The network oscillators consisting of oscillators (1) coupled by connection (2) have a steady state,

$$\left[\boldsymbol{x}_1^T \cdots \boldsymbol{x}_N^T\right]^T = \left[\boldsymbol{x}^{*T} \cdots \boldsymbol{x}^{*T}\right]^T. \tag{4}$$

The dynamics around steady state (4) is governed by

$$\left\{ \begin{array}{l} \Delta\dot{\boldsymbol{x}}_i = \boldsymbol{A}\Delta\boldsymbol{x}_i + \boldsymbol{b}\Delta\boldsymbol{u}_i \\ \Delta\boldsymbol{y}_i = \boldsymbol{c}\Delta\boldsymbol{x}_i \end{array} \right. , \tag{5}$$

where $\Delta\boldsymbol{x}_i$, $\Delta\boldsymbol{y}_i$, and $\Delta\boldsymbol{u}_i$ are the perturbations of oscillator i around $\boldsymbol{x}^*$. Here $\boldsymbol{A} := \{\partial\boldsymbol{F}/\partial\boldsymbol{x}\}_{\boldsymbol{x}=\boldsymbol{x}^*}$ is the Jacobian matrix. The coupling signal is given by

$$\Delta\boldsymbol{u}_i = \boldsymbol{k}_s \left\{ \frac{1}{d_i} \left(\sum_{l=1}^{N} \varepsilon_{il} \Delta\boldsymbol{y}_{l,\tau} \right) - \Delta\boldsymbol{y}_i \right\}. \tag{6}$$

The linear system consisting of Eq. (5) and Eq. (6) is described by

$$\dot{\boldsymbol{X}} = (\boldsymbol{I}_N \otimes \boldsymbol{A}_s)\boldsymbol{X} + \boldsymbol{k}_s(\boldsymbol{E} \otimes \boldsymbol{bc})\boldsymbol{X}_\tau, \tag{7}$$

where $\boldsymbol{X} := [\Delta\boldsymbol{x}_1^T \cdots \Delta\boldsymbol{x}_N^T]^T$, $\boldsymbol{X}_\tau := \boldsymbol{X}(t - \tau(t))$, and $\boldsymbol{A}_s := \boldsymbol{A} - \boldsymbol{k}_s\boldsymbol{bc}$. The matrix $\boldsymbol{I}_N$ denotes a N-dimensional identity matrix. The elements of $\boldsymbol{E}$ are given by $\{\boldsymbol{E}\}_{il} = \varepsilon_{il}/d_l$ $(l \neq i)$ and $\{\boldsymbol{E}\}_{ii} = 0$.

3. Stability analysis

It is guaranteed that the time-invariant system,

$$\dot{\boldsymbol{X}} = (\boldsymbol{I}_N \otimes \boldsymbol{A}_s)\boldsymbol{X} + \frac{1}{2\delta}\boldsymbol{k}_s(\boldsymbol{E} \otimes \boldsymbol{bc}) \int_{t-\tau_0-\delta}^{t-\tau_0+\delta} \boldsymbol{X}(z)\mathrm{d}z, \tag{8}$$

is stable if and only if linear system (7) with large Ω is stable [14]. The stability of time-invariant system (8) is described by the characteristic equation $G(s) = 0$, where

$$G(s) := \det\left[s\boldsymbol{I}_{Nm_x} - \boldsymbol{I}_N \otimes \boldsymbol{A}_s - \boldsymbol{k}_s(\boldsymbol{E} \otimes \boldsymbol{bc})e^{-s\tau_0}H(s\delta)\right]. \tag{9}$$

The function $H(x)$ is defined by

$$H(x) := \begin{cases} \frac{\sinh x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}. \tag{10}$$

Since $(\boldsymbol{I}_N - \boldsymbol{E})$ is a self-adjoint and positive semidefinite operator [15], it can be diagonalized as $\boldsymbol{T}^{-1}(\boldsymbol{I}_N - \boldsymbol{E})\boldsymbol{T} = \mathrm{diag}(\rho_1, \ldots, \rho_N)$, where $\boldsymbol{T}$ is a diagonal transmission matrix and $\rho_p (p = 1, \ldots, N)$ is an eigenvalue of $(\boldsymbol{I}_N - \boldsymbol{E})$. Thus, we can rewrite Eq. (9) as follows:

$$\begin{aligned} G(s) = \det\Big[&(\boldsymbol{T}^{-1} \otimes \boldsymbol{I}_{m_x}) \\ &\{s\boldsymbol{I}_{Nm_x} - \boldsymbol{I}_N \otimes \boldsymbol{A}_s - \boldsymbol{k}_s(\boldsymbol{E} \otimes \boldsymbol{bc})e^{-s\tau_0}H(s\delta)\} \\ &(\boldsymbol{T} \otimes \boldsymbol{I}_{m_x})\Big] \\ = \det\Big[&s\boldsymbol{I}_{Nm_x} - \boldsymbol{I}_N \otimes \boldsymbol{A}_s \\ &-\boldsymbol{k}_s\left(\boldsymbol{I}_N - \boldsymbol{T}^{-1}(\boldsymbol{I}_N - \boldsymbol{E})\boldsymbol{T} \otimes \boldsymbol{bc}\right)e^{-s\tau_0}H(s\delta)\Big] \\ = \det\Big[&s\boldsymbol{I}_{Nm_x} - \boldsymbol{I}_N \otimes \boldsymbol{A}_s \\ &-\boldsymbol{k}_s\left(\boldsymbol{I}_N - \mathrm{diag}(\rho_1, \ldots, \rho_N) \otimes \boldsymbol{bc}\right)e^{-s\tau_0}H(s\delta)\Big]. \end{aligned}$$

The characteristic equation can be described by

$$G(s) = \prod_{p=1}^{N} g_s(s, \rho_p) = 0, \tag{11}$$

where

$$g_s(s, \rho_p) := \det\left[s\boldsymbol{I}_{m_x} - \boldsymbol{A}_s - \boldsymbol{k}_s(1 - \rho_p)\boldsymbol{bc}e^{-s\tau_0}H(s\delta)\right], \quad (p = 1, 2, \ldots, N). \tag{12}$$

The time-invariant system (8) is stable if and only if all the roots of $g_s(s, \rho_p) = 0$ for all $p \in \{1, 2, \ldots, N\}$ lie in the left half complex plane.

The eigenvalue of $(\boldsymbol{I}_N - \boldsymbol{E})$ is given by

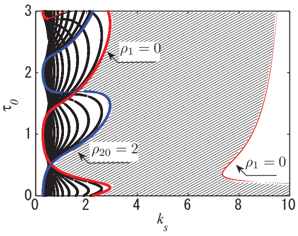
$$0 = \rho_1 \leq \rho_2 \leq \cdots \leq \rho_N \leq 2. \tag{13}$$

Since the minimum eigenvalue ρ_1 is fixed at zero, Eq. (11) includes

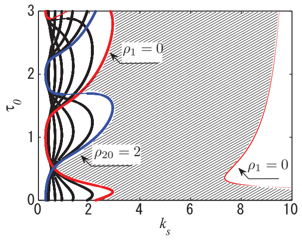
$$g_s(s, 0) = \det\left[s\boldsymbol{I}_{m_x} - \boldsymbol{A}_s - \boldsymbol{k}_s\boldsymbol{bc}e^{-s\tau_0}H(s\delta)\right]. \tag{14}$$

Now we derive the sufficient condition in which amplitude death never occurs. For $s \in \mathbb{R}$, we have

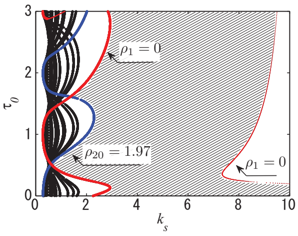
$$\lim_{s \to +\infty} g_s(s, 0) = +\infty. \tag{15}$$



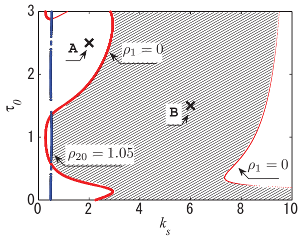
(a) chain



(b) ring



(c) small-world ($N_c = 5$)



(d) all-to-all

Figure 3: Stability regions for various network topologies ($\delta = 0.38$, $N = 20$)

Substituting $s = 0$ into $g_s(s, \rho_1)$, we obtain

$$g_s(0, 0) = \det\left[-\boldsymbol{A}\right] = \prod_{q=1}^{m_x}(-\sigma_q),$$

where $\sigma_1, \ldots, \sigma_{m_x}$ are the eigenvalues of $\boldsymbol{A}$. If $\boldsymbol{A}$ has an odd-number of real positive eigenvalues, $g_s(0, 0) < 0$ holds. For $g_s(0, 0) < 0$ and Eq. (15), equation $g_s(s, 0) = 0$ has at least one real positive root; therefore, if $\boldsymbol{A}$ has an odd-number of real positive eigenvalues, amplitude death never occurs for any $\boldsymbol{E}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{k}_s$, τ_0, and δ.

4. Limit cycle oscillators

Let us consider oscillators (1) and connection (2) with

$$\boldsymbol{F}(\boldsymbol{x}) = \begin{bmatrix} \left\{\mu - x(1)^2 - x(2)^2\right\} x(1) - \omega x(2) \\ \left\{\mu - x(1)^2 - x(2)^2\right\} x(2) + \omega x(1) \end{bmatrix}, \tag{16}$$

$$\boldsymbol{b} = \boldsymbol{I}_2, \ \boldsymbol{c} = \boldsymbol{I}_2, \ \boldsymbol{k}_s = k_s \boldsymbol{I}_2, \tag{17}$$

where $\mu > 0$ and $\omega > 0$ denote the amplitude and the frequency of the oscillator, respectively. Each oscillator has an unstable fixed point $\boldsymbol{x}^* = [0\ 0]^{\mathrm{T}}$. The Jacobian matrix around the fixed point is given by

$$\boldsymbol{A} = \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix}. \tag{18}$$

The eigenvalues $\sigma_{1,2} = \mu \pm j\omega$ of $\boldsymbol{A}$, which have two real positive roots, do not satisfy the above condition for non-occurrence of amplitude death. The characteristic equation (12) is described by

$$g_s(s, \rho_p) = \left[s - \mu + k_s \left\{1 - (1 - \rho_p) e^{-s\tau_0} H(s\delta)\right\}\right]^2 + \omega^2. \tag{19}$$

Substituting $s = j\lambda$, $\lambda \in \mathbb{R}$, into $g_s(s, \rho_p) = 0$, its real and imaginary parts are given by

$$-\mu + k_s - k_s(1 - \rho_p)\phi(\lambda\delta)\cos(\lambda\tau_0) = 0, \tag{20}$$

$$\lambda - \omega + k_s(1 - \rho_p)\phi(\lambda\delta)\sin(\lambda\tau_0) = 0, \tag{21}$$

where

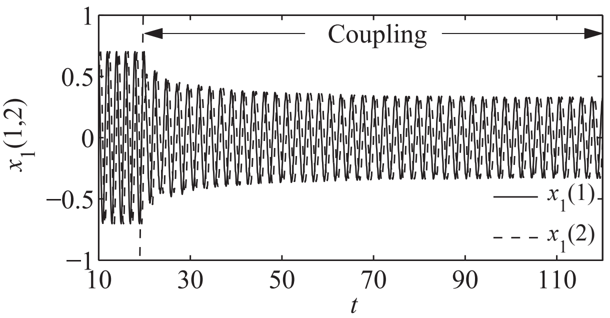
$$\phi(x) := \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}. \tag{22}$$

According to the analysis of previous study [13], we can derive the marginal stability curves in (k_s, τ_0) space.

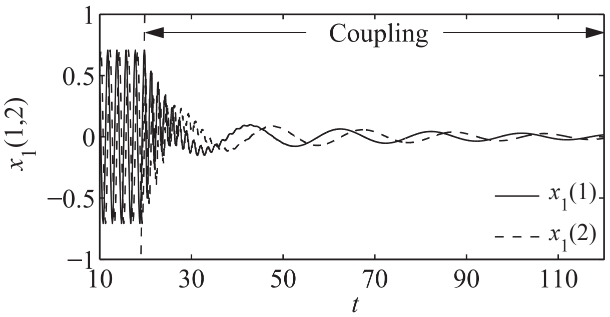
Now we consider numerical examples for the delay amplitude $\delta = 0.38$ and the number of oscillators $N = 20$. The parameters of oscillators are set to $\mu = 0.5$ and $\omega = \pi$. Figure 3 shows the stability regions for various network topologies: chain, ring, small-world (the number of shortcuts $N_C = 5$), and all-to-all. The black, red, and blue curves denote the marginal stability curves which are the solution of $g_s(j\lambda, \rho_p) = 0$ in terms of k_s and τ_0. For a given τ_0, when k_s increases and crosses the bold (thin) curve, a root of $g_s(s, \rho_p) = 0$ crosses the imaginary axis from right to left (left to right). The shaded area denotes the stability region where all the roots of $g_s(s, \rho_p) = 0$, $\forall p \in \{1, \ldots, 20\}$, lie in the left half complex plane. For all the network topologies, there exist the stability regions which have no upper limit in the range $k_s \in (2.94, 7.34)$. Thus, steady state (4) can be stabilized by the arbitrarily long delay τ_0. The red and blue curves denote the solution of $g(j\lambda, \rho_1) = 0$ and $g(j\lambda, \rho_{20}) = 0$. It can be seen that the stability regions are surrounded by these two curves. Hence, it could be considered that the stability region depends on the curves with the minimum $\rho_1 = 0$ and the maximum ρ_N.

For both of the chain topology and the ring topology, their maximum eigenvalues of $(\boldsymbol{I}_N - \boldsymbol{E})$ are $\rho_{20} = 2$ and their stability regions are identical. For the all-to-all topology, the maximum $\rho_{20} = 1.05$ is the lowest among all the topologies and the stability region is the largest among them. Therefore, we can see that the stability region in (k_s, τ_0) space enlarges with decreasing the maximum eigenvalue ρ_N.

To confirm our analytical results, we show the numerical simulations. Figure 4 shows time-series data at point A $(k_s, \tau_0) = (2.0, 2.5)$ and point B $(k_s, \tau_0) = (6.0, 1.5)$ in Fig. 3(d). The solid and dash lines are the behavior of $x_1(1)$ and $x_1(2)$ of the 1st oscillator, respectively. All the oscillators are coupled at $t = 20$. At point A outside of the stability region, $x_1(1)$ and $x_1(2)$ do not converge on the fixed point $\boldsymbol{x}^*$ after the coupling. In contrast, at point B inside of the stability region, they converge on the fixed point $\boldsymbol{x}^*$ after the coupling. It is numerically confirmed that amplitude death occurs in network oscillators by using the time-varying delay connection.



(a) Point A $(k_s, \tau_0) = (2.0, 2.5)$



(b) Point B $(k_s, \tau_0) = (6.0, 1.5)$

Figure 4: Time-series data in Fig. 3(d)

5. Conclusion

The present paper showed that amplitude death occurs in network oscillators by using the time-varying delay connection. For the various network topologies, we provided the stability regions in the connection parameter space. These results are confirmed by the numerical simulations.

Acknowledgments

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