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Ginzburg-Landau Equations
Reduced from Coupled Delay Differential Equations

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Abstract—A coupled system of delay differential equations motivated through the corticothalamic dynamics is systematically reduced to the complex and the real Ginzburg-Landau equations. Two analytically solvable problems are discussed; (1) relaxation process from any initial state to the corresponding attractor (limit cycles or attractive fixed point) and (2) amplitude death of coupled two oscillators due to "negative average bifurcation parameter" as well as due to large frequency difference. Projection from infinite dimensional phase space to the center subspace (subspace spanned by eigenfunctions belonging to zero or pure imaginary eigenvalues) plays mathematically essential role through this study and is clearly illustrated in (1). Physiologically, on the other hand, (2) provides unique insights into observed EEG activities such as occurrence of epileptic seizure.

1. Introduction

Although time delay plays essential role in many systems [1], its analysis has been limited because of its infinite dimensionality [2]. If we direct our attention to the vicinity of bifurcations, however, simplified equations which represent essential phenomena are expected to be derived because damped modes are enslaved to excited modes and can thus be "adiabatically eliminated" in such parameter regions. The center manifold theory, as is well known, realizes this idea mathematically [3]. We have proposed the reduction method [4] based on the center manifold theory and perturbation theory [5] in the vicinity of Hopf bifurcation point with a corticothalamic model [6] as a sample model. Well-known complex Ginzburg-Landau equation of discrete version was derived from the original model.

In this paper, after a brief review of our reduction method in somewhat generalized form including pitchfork bifurcation, two analytically solvable problems are demonstrated; (1) relaxation processes from any initial conditions to the attractors and (2) amplitude death of coupled two oscillators due to "negative average bifurcation parameter" as well as large frequency difference.

2. Sample model to be reduced

In the present study, we analyze the following equations originated from Kim and Robinson's corticothalamic model [6]:

$$\begin{aligned}\ddot{x}_j(t) &= \gamma\dot{x}_j(t)+\alpha_j x_j(t)+\beta_j x_j(t-t_0)+\epsilon x_j(t)^3\\ &\quad+\sum_{k=1}^{N}K_{jk}(x_k(t)-x_j(t)),\end{aligned}\tag{1}$$

where $x_j(t)$ is a mean firing rate of neurons within each local area (denoted by $j = 1, \cdots, N$) of the cortex; t_0 is the time delay; α_j parameterizes the strength of cortico-cortical activities; β_j characterizes corticothalamic feedback; γ gives the damping rate; ϵ control the nonlinear terms that are originated from the characteristics of neuronal firing; and $\sum_{k=1}^{N} K_{jk}(x_k(t) - x_j(t))$ represents interaction between local areas j and k. In this paper, however, we regard Eq. (1) one of typical coupled delay equations and don't go into detail about its physiological view. All through this paper, we take α_j and β_j as variable parameters, while the other parameters are fixed at $\gamma = -2$, $\epsilon = -10$, and $t_0 = 8$.

We define A_j and $\mu_j \equiv \alpha_j - A_j$ such that A_j gives a critical value of α_j in a bifurcation (destabilization of the equilibrium point $x_j(t) \equiv 0$) under the condition $K_{jk} = 0$. A_j is determined with the characteristic equation

$$C_h(\mu_j,\lambda) \equiv \lambda^2-\gamma\lambda-(A_j+\mu_j)-\beta_j e^{-\lambda t_0} = 0. \tag{2}$$

$C_h(0, i\Omega_j) = 0$ determines A_j and pure imaginary eigenvalue $i\Omega_j$ at the Hopf bifurcation point, while $C_h(0, 0) = 0$ determines A_j at the pitchfork bifurcation point.

To extend the center manifold reduction to the neighborhood of the bifurcation point, we define three dynamical variables (x_j^{1-3}) including the bifurcation parameter μ_j as $x_j(t) = x_j^1(t)$, $\dot{x}_j(t) = x_j^2(t)$, $\mu_j = -\epsilon(x_j^3(t))^2$, and rewrite Eq. (1) in matrix representation as below.

$$\frac{d}{dt}\begin{bmatrix} x_j^1(t)\\ x_j^2(t)\\ x_j^3(t)\end{bmatrix} = A_j^0\begin{bmatrix} x_j^1(t)\\ x_j^2(t)\\ x_j^3(t)\end{bmatrix} + A_j^1\begin{bmatrix} x_j^1(t-t_0)\\ x_j^2(t-t_0)\\ x_j^3(t-t_0)\end{bmatrix}$$

$$+\mathbf{F}(\mathbf{x}_j(t)) + \sum_{k=1}^{N} K_{jk}\mathbf{G}(\mathbf{x}_k(t) - \mathbf{x}_j(t)), \quad (3)$$

$$A_j^0 = \begin{bmatrix} 0 & 1 & 0 \\ A_j & \gamma & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; A_j^1 = \begin{bmatrix} 0 & 0 & 0 \\ \beta_j & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (4)$$

$$\mathbf{F}(\mathbf{x}_j(t)) = \begin{bmatrix} 0 \\ \epsilon x_j^1(t)[-(x_j^3(t))^2 + (x_j^1(t))^2] \\ 0 \end{bmatrix}, \quad (5)$$

$$\mathbf{G}(\mathbf{x}_k(t) - \mathbf{x}_j(t)) = \begin{bmatrix} 0 \\ x_k^1(t) - x_j^1(t) \\ 0 \end{bmatrix}. \quad (6)$$

3. Reduction to Ginzburg-Landau equations

3.1. Basis and dual basis of the center subspace

We consider the following functional differential equation:

$$\frac{d}{dt}\mathbf{x}^{(t)}(\eta) = \begin{cases} \frac{d}{d\eta}\mathbf{x}^{(t)}(\eta) & (-t_0 < \eta < 0) \\ A^0\mathbf{x}^{(t)}(0) + A^1\mathbf{x}^{(t)}(-t_0) & (\eta = 0) \end{cases} \quad (7)$$

This equation corresponds to the linear noninteracting part of Eq. (3), index $_j$ omitted. $\mathbf{x}^{(t)}(\eta) \equiv \mathbf{x}(t+\eta), \eta \in [-t_0, 0]$ represents a state function in a infinite dimensional phase space $C([-t_0, 0] \to R^3)$. Ansatz $\mathbf{x}^{(t)}(\eta) = e^{\lambda t}\phi(\eta)$ leads the following equations:

$$[\lambda I - A^0 - e^{-\lambda t_0}A^1]\phi(0) = 0, \; \frac{d\phi(\eta)}{d\eta} = \lambda\phi(\eta). \quad (8)$$

The base of the center subspace is obtained from this equation with the associated eigenvalue λ. A dual base ψ is obtained as the adjoint problem of Eq. (8), that is,

$$\psi(0)[\lambda I - A^0 - e^{-\lambda t_0}A^1] = 0, \; \frac{d\psi}{d\xi} = -\lambda\psi(\xi). \quad (9)$$

We can show that $\det[\lambda I - A^0 - e^{-\lambda t_0}A^1] = C_h(0, \lambda)$ and therefore, Eq. (8) or Eq. (9) have nontrivial solutions with the associated eigenvalue λ from our definition of A.

Conventionally, the basis and dual basis are writen in the matrix form as

$$\Phi \equiv [\phi_1 \; \phi_2 \; \ldots \; \phi_n], \quad \Psi \equiv \begin{bmatrix} \psi^1 \\ \psi^2 \\ \vdots \\ \psi^n \end{bmatrix}, \quad (10)$$

where n denotes the dimension of the center subspace. We determine Φ as $\Phi_1^1(0) = 1$ and $\Phi_n^a = \delta_n^a$ (Kronecker delta) and normalize Ψ as $\langle\psi^a, \phi_b\rangle = \delta_b^a$, where

$$\langle\psi, \phi\rangle \equiv \psi(0)\phi(0) + \int_{-t_0}^{0} \psi(\sigma + t_0)A^1\phi(\sigma)d\sigma \quad (11)$$

represents the duality between the center subspace and its dual space. In the case of the Hopf bifurcation, where a pair of complex conjugate eigenfunctions $\phi_\pm$ and their dual $\psi^\pm$ appear belonging to pure imaginary eigenvalues $\pm i\Omega$, we transform basis and dual basis to the real Jordan normal form; $\phi_1 \equiv Re(\phi_+)$, $\phi_2 \equiv Im(\phi_+)$, $\psi^1 \equiv 2Re(\psi^+)$, $\psi^2 \equiv -2Im(\psi^+)$.

3.2. General form of the reduced equations

State function $\mathbf{x}_j^{(t)}$ can be decomposed to the center subspace component $\mathbf{u}_j(t)$ and its opponent $\mathbf{h}_j$ as below.

$$\mathbf{x}_j^{(t)}(\eta) = \Phi_j(\eta)\mathbf{u}_j(t) + \mathbf{h}_j, \quad \mathbf{u}_j(t) = \langle\Psi_j(\eta), \mathbf{x}_j^{(t)}(\eta)\rangle \quad (12)$$

In the vicinity of bifurcation point, the steady (not transient) system state $\mathbf{x}_j^{(t)}$ is so close to the center subspace that $\mathbf{h}_j$ would be negligible. The Reduced equations under this assumption are obtained in the form of

$$\frac{d}{dt}\mathbf{u}_j(t) = B_j\mathbf{u}_j(t) + \mathbf{N}_j + \mathbf{I}_j, \quad (13)$$

$$\mathbf{N}_j = \Psi_j(0)F_j\left(\Phi_j(0)\mathbf{u}_j(t)\right), \quad (14)$$

$$\mathbf{I}_j = \Psi_j(0)\sum_{k=1}^{N} G_{jk}\left(\Phi_k(0)\mathbf{u}_k(t) - \Phi_j(0)\mathbf{u}_j(t)\right), \quad (15)$$

where matrix B_j has the Jordan normal form satisfying the following equations:

$$\frac{d\Phi_j}{d\eta} = \Phi_j B_j, \quad \frac{d\Psi_j}{d\xi} = -B_j\psi_j. \quad (16)$$

3.3. Complex Ginzburg-Landau equation

Applying the above equations for the Hopf bifurcation ($\lambda = \pm i\Omega$), $\Phi_j(\eta)$, $\Psi_j(\xi)$ and B_j are obtained as below.

$$\Phi_j(\eta) = \begin{bmatrix} \cos\Omega_j\eta & \sin\Omega_j\eta & 0 \\ -\Omega_j\sin\Omega_j\eta & \Omega_j\cos\Omega_j\eta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (17)$$

$$\begin{aligned}
\Psi_j(\xi)_1^1 &= (a_j\Omega_j + b_j\gamma)\sin\Omega_j\xi + (-a_j\gamma + b_j\Omega_j)\cos\Omega_j\xi \\
\Psi_j(\xi)_2^1 &= -b_j\sin\Omega_j\xi + a_j\cos\Omega_j\xi \\
\Psi_j(\xi)_1^2 &= (b_j\Omega_j - a_j\gamma)\sin\Omega_j\xi - (b_j\gamma + a_j\Omega_j)\cos\Omega_j\xi \\
\Psi_j(\xi)_2^2 &= a_j\sin\Omega_j\xi + b_j\cos\Omega_j\xi \\
\Psi_j(\xi)_3^1 &= \Psi_j(\xi)_3^2 = \Psi_j(\xi)_1^3 = \Psi_j(\xi)_2^3 = 0, \; \Psi_j(\xi)_3^3 = 1,
\end{aligned} \quad (18)$$

where $a_j = l_j/\{(l_j)^2 + (m_j)^2\}$, $b_j = m_j/\{(l_j)^2 + (m_j)^2\}$ with $l_j = (-\gamma + \beta_j t_0\cos\Omega_j t_0)/2$, $m_j = (2\Omega_j - \beta_j t_0\sin\Omega_j t_0)/2$, and

$$B = \begin{bmatrix} 0 & \Omega_j & 0 \\ -\Omega_j & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (19)$$

Using these results and returning to μ_j again from x_j^3, we arrive at the following reduced equation:

$$\begin{bmatrix} \dot{u}_j^1(t) \\ \dot{u}_j^2(t) \end{bmatrix} = \begin{bmatrix} \mu_j a_j & \Omega_j \\ -\Omega_j + \mu_j b_j & 0 \end{bmatrix} \begin{bmatrix} u_j^1 \\ u_j^2 \end{bmatrix} + \left[\epsilon (u_j^1)^3 + \sum_{k=1}^{N} K_{jk}(u_k^1 - u_j^1) \right] \begin{bmatrix} a_j \\ b_j \end{bmatrix} \quad (20)$$

Furthermore, if we focus on the situation that Ω_j are narrowly distributed, we can derive a further simplified equation from Eq. (1) by the averaging method [3] as

$$\dot{u}_j(t) = -i\Omega_j u_j + c_j \left(\frac{1}{2}\mu_j + \frac{3}{8}\epsilon |u_j|^2 \right) u_j + I_j, \quad (21)$$

$$I_j = \frac{1}{2} c_j \sum_{k=1}^{N} K_{jk} \left(u_k - u_j \right), \quad (22)$$

where we introduced a complex variable $u_j = u_j^1 + iu_j^2$ and $c_j = a_j + ib_j$.

3.4. Real Ginzburg-Landau equation

In the case of the pitchfork bifurcation ($\lambda = 0$), we fined $B_j = 0$ and

$$\Phi_j(\xi) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad (23)$$

$$\Psi_j(\eta) = \begin{bmatrix} -\gamma/(-\gamma + \beta_j t_0) & 1/(-\gamma + \beta_j t_0) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (24)$$

With these results, we arrive at the bellow reduced equation.

$$\dot{u}_j(t) = \frac{\epsilon_j}{-\gamma_j + \beta_j t_0} \left\{ (u_j)^2 - (\frac{\mu_j}{-\epsilon_j}) \right\} u_j + I_j, \quad (25)$$

$$I_j = \frac{1}{-\gamma_j + \beta_j t_0} \sum_{k=1}^{N} K_{jk} \left(u_k - u_j \right). \quad (26)$$

4. Relaxation process

The reduced equations elucidate some essential properties involved in the original model in an analytical manner. For the first example, let us discuss the relaxation process using (1)cosine function $x^{(0)}(\eta) = \cos(\Omega\eta)$, (2)Heaviside function $x^{(0)}(\eta) = 0, \eta < 0$, $x^{(0)}(0) = 1$, and (3)constant function $x^{(0)}(\eta) = 1$ as initial functions. Note that x is the same for all the case at $t = 0$, but different in the past ($t < 0$). Therefore, relaxation process varies case by case.

4.1. To limit cycle

Using Eq. (21) with I=0, amplitude $|u(t)|$ can be obtained as

$$|u(t)| = |u(0)| \left[e^{-\frac{t}{\tau}} \{ 1 - (\frac{|u(0)|}{|u(\infty)|})^2 \} + (\frac{|u(0)|}{|u(\infty)|})^2 \right]^{-\frac{1}{2}}. \quad (27)$$

where $|u(\infty)| = \sqrt{4\mu/(-3\epsilon)}$ and $\tau = (1/2)(-\gamma + \beta t_0)/\mu$. $u(0) = \langle \psi^1, \mathbf{x}^{(0)} \rangle + i \langle \psi^2, \mathbf{x}^{(0)} \rangle$ turn out to be 1 (cosine), $-(a + ib)$ (Heaviside), $u(0) = i(a + ib)(A + \beta)/\Omega$ (constant).

Fig.1 shows the time evolution calculated with the original equation Eq. (1) (blue line) and the predicted amplitude calculated with Eq. (27) (red line), in the case of cosine initial function (Top) and Heaviside initial function (Bottom). The right side panels show the expanded view of the same results as the left side ones. $\beta = -1.8$, $\mu = 0.012$ are used and initial functions are normalized as $|u(0)| = 0.00492$. Difference between the analytical prediction and the numerical result are immediately damped. Therefore, our prediction is still effective as a whole. Note that an appropriate prediction cannot be obtained without the use of the projection $u(0) = \langle \psi^1, \mathbf{x}^{(0)} \rangle + i \langle \psi^2, \mathbf{x}^{(0)} \rangle$.

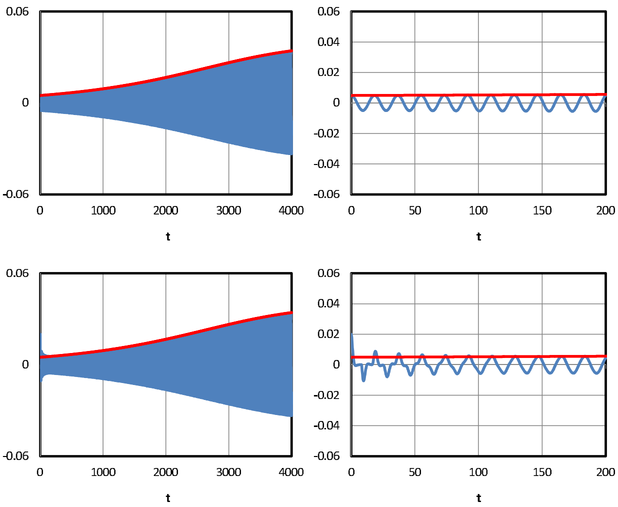

Figure 1: Relaxation Process.The time evolution calculated with the original equation Eq. (1) (blue line) and the predicted amplitude calculated with Eq. (27) (red line), in the case of cosine initial function (Top) and Heaviside initial function (Bottom). The right side panels show the expanded view of the same results as the left side ones.

4.2. To attractive fixed point

Eq. (25) with I=0 is integrated as

$$u(t) = u(0) \left[e^{-\frac{t}{\tau}} \{ 1 - (\frac{u(0)}{u(\infty)})^2 \} + (\frac{u(0)}{u(\infty)})^2 \right]^{-\frac{1}{2}}, \quad (28)$$

where $u(\infty) = \sqrt{\mu/(-\epsilon)}$, $\tau = (1/2)(-\gamma + \beta t_0)/\mu$. $u(0) = \langle \psi^1, \mathbf{x}^{(0)} \rangle$ results in $(-\gamma + \beta \sin \omega t_0/\omega)/(-\gamma + \beta t_0)$ (cosine), $\gamma/(\gamma - \beta t_0)$ (Heaviside), 1 (constant).

5. Amplitude death of 2-Body problem

5.1. Negative average μ

Let us consider the 2-Body problem of Eq. (21) in the case of $\Omega_1 = \Omega_2 = \Omega$. The matrix corresponding to the linear part of Eq. (21) under this condition results in

$$M = \begin{bmatrix} -i\Omega + \frac{1}{2}c(\mu_1 - K) & \frac{1}{2}cK \\ \frac{1}{2}cK & -i\Omega + \frac{1}{2}c(\mu_2 - K) \end{bmatrix}. \quad (29)$$

The following inequalities must hold if two eigenvalues of M have both negative real part:

$$\mu_1 + \mu_2 < 0, \quad K > \frac{\mu_1\mu_2}{\mu_1 + \mu_2}. \quad (30)$$

Fig.2 illustrates this analytical prediction by numerical calculation of Eq. (1). $\Omega = 0.25$ is used.

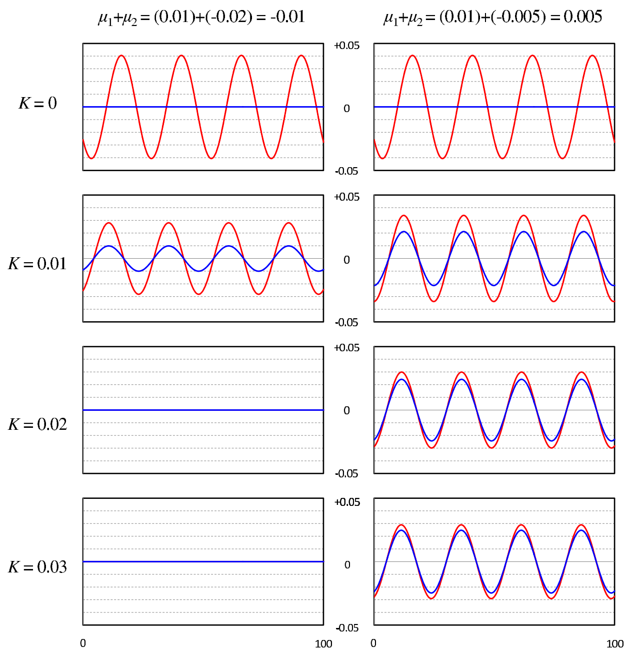

Figure 2: Amplitude death due to negative average μ in 2-body problem. Sufficiently strong connection such that $K > \mu_1\mu_2/(\mu_1 + \mu_2)$ suppresses oscillation if $(\mu_1 + \mu_2)$ is negative (left), but doesn't positive (right).

5.2. Large frequency difference

Let us consider the case of $\mu_1 = \mu_2 = \mu$ in turn. Amplitude death would occur when difference between Ω_1 and Ω_2 is sufficiently large. The relevant matrix can be decomposed as $M = M_0 + \Delta M$ with

$$M_0 = \begin{bmatrix} 0 & \Omega_j & 0 & 0 \\ -\Omega_j & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega_k \\ 0 & 0 & -\Omega_k & 0 \end{bmatrix}, \quad (31)$$

$$\Delta M = \begin{bmatrix} a_j(\mu - K) & 0 & a_jK & 0 \\ b_j(\mu - K) & 0 & b_jK & 0 \\ a_kK & 0 & a_k(\mu - K) & 0 \\ b_kK & 0 & b_k(\mu - K) & 0 \end{bmatrix}. \quad (32)$$

The right eigenvectors of M_0 are

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ i & -i & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & i & -i \end{bmatrix}, \quad (33)$$

and the left eigenvectors U^* are just Hermite conjugate of U. The exact calculation for eigenvalues of M is complicated in this case, but the perturbed eigenvalues of M can be obtained from the diagonal components of U^*MU as below.

$$\lambda_j \simeq \frac{a_j}{2}(\mu - K) \pm i\left[\Omega_j - \frac{b_j}{2}(\mu - K)\right] \quad (j = 1, 2). \quad (34)$$

Hence, if K exceeds μ, the real part of λ_j becomes negative, which implies the possibility of the amplitude death phenomenon. See [4] for its numerical confirmation.

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