IEICE Proceeding Series

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Vol. 2 pp. 90-93
Publication Date: 2014/03/18
Online ISSN: 2188-5079

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Abstract—In the paper, a method of finite settling time control of constrained systems is described. The method is an extension of the relatively optimal control scheme by Blanchini and Pellegrino, and can derive larger attractive regions and faster convergence to the equilibrium point. Usefulness of the proposed method are demonstrated through some examples.

1. Introduction

For almost all practical control systems, we need to take into account constraints on state and/or control input caused by amplitude limitation of state variables, saturation property of actuators and so on. If we ignore these constraints, then the real performance of the system degrades or, in worst cases, the control system becomes unstable. In these respect, extensive researches have been done to cope with such constraints (See [1] –[5] and references therein).

In this paper, we consider a state feedback dead-beat control and will propose an extension of the static ROC (Relatively Optimal Control) method [6]. The proposing method has two advantages: The first is that the attractive region is larger than that obtained by ROC. The second is achieving faster convergence to the origin than the ROC does.

2. Problem statement

2.1. System Description

Consider a discrete-time system given by

$$\begin{cases} x[k+1] = Ax[k] + bu[k], \quad x[0] = x_0, \\ z[k] = Lx[k] + Du[k], \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state of the system, $x_0 \in \mathbb{R}^n$ is the initial state, $u \in \mathbb{R}$ is the control input, and $z \in \mathbb{R}^m$ is the vector of constrained variables. We assume that A is nonsingular and (A, b) is a reachable pair.

When x_0 and $\boldsymbol{u}_k = \begin{bmatrix} u[0] & u[1] & \cdots & u[k-1] \end{bmatrix}^\top$ are given, $x[k; x_0, \boldsymbol{u}_k]$ denotes the solution of the system (1). The constraint is represented by $z[k; x_0, \boldsymbol{u}_{k+1}] = (Lx[k; x_0, \boldsymbol{u}_k] + Du[k]) \in \mathcal{Z}$ for all $k \geq 0$, where $\boldsymbol{u}_{k+1} = \begin{bmatrix} \boldsymbol{u}_k^\top & u[k] \end{bmatrix}^\top$, and $\mathcal{Z}$ is a polytope given by

$$\mathcal{Z} = \{z : \ Hz \leq h\}, \tag{2}$$

where $H \in \mathbb{R}^{N_c \times m}$, $h \in \mathbb{R}^{N_c}$, $h > 0$, and inequalities $\leq$ and $>$ means the element-wise inequalities.

We say that a region $\mathcal{X}_q$ is a q-time attractive region if $x_0 \in \mathcal{X}_q$ then there exists an input $\boldsymbol{u}_q$ such that $z[k; x_0, \boldsymbol{u}_q] \in \mathcal{Z}$ for all $k = 0, 1, \cdots q-1$ and $x[q; x_0, \boldsymbol{u}_q] = 0$.

2.2. An Motivative Example

Blanchini and Pellegrino [6] proposed the static ROC. For a given positive integer N and an initial state x_0, they consider the following optimization problem.

$$\text{(QP1)} \begin{cases} \displaystyle\min_{x[\cdot],u[\cdot],z[\cdot]} & \displaystyle\sum_{k=0}^{N-1} \left(|Cx[k]|^2 + R|u[k]|^2\right) \\ \text{sub. to} & x[k+1] = Ax[k] + bu[k], \\ & z[k] = Lx[k] + Du[k] \in \mathcal{Z}, \\ & \qquad k = 0, 1, \cdots, N-1 \\ & x[0] = x_0, \quad x[N] = 0 \end{cases}$$

where $C \in \mathbb{R}^{1 \times n}$ and R is a positive number.

Using the optimal solution $\{(\hat{x}[k+1], \ \hat{u}[k], \hat{z}[k])\}_{k=0}^{N-1}$, they construct attractive regions $\{\mathcal{X}_q(x_0)\}_{q=1}^{N}$ (in this case, $\mathcal{X}_q$ depends on x_0).

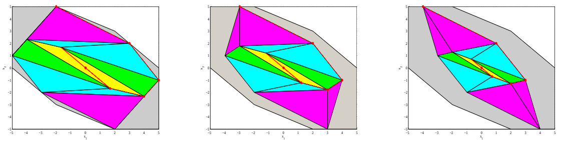
Example 1 *Let $C = 0$, $R = 1$ in (QP1), and*

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{3}$$

$$H = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \quad h = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \\ 3 \\ 3 \end{bmatrix}. \tag{4}$$

For $q = 1, 2, 3, 4, N$, $N = 5$ and $x_0 = [-2 \ -5]^\top$, $x_0 = [-3 \ -5]^\top$, and $x_0 = [-4 \ -5]^\top$, we compute $\mathcal{X}_q(x_0)$. In Fig. 1, we show $\{\mathcal{X}_q(x_0)\}_{q=1}^{N}$, where the red, the yellow, the green, the cyan, and the magenta regions denotes $\mathcal{X}_1(x_0)$, $\mathcal{X}_2(x_0)$, $\mathcal{X}_3(x_0)$, $\mathcal{X}_4(x_0)$, and $\mathcal{X}_5(x_0)$ respectively. About the gray region we will mention it later.

As we can see from **Figure 1**, attractive regions $\mathcal{X}_q(x_0)$ depend on initial state x_0. Note that $\tilde{x}_0$



$x_0 = [-2 \ -5]^\top$ $x_0 = [-3 \ -5]^\top$ $x_0 = [-4 \ -5]^\top$

Figure 1: Attractive regions $\{\mathcal{X}_q(x_0)\}_{q=1}^5$ computed by ROC method.

may belong to $\mathcal{X}_q(x_0) \cap \mathcal{X}_{q'}(x_0')$ for different (q, x_0) and (q', x_0'). In this case, we have two control laws $u = f(x)$ and $u = f'(x)$, and $x[q] = 0$ if we adopt $u[k] = f(x[k])$ and $x[q'] = 0$ if we adopt $u[k] = f'(x[k])$. This implies that the control law $u = f(x)$ is not the control law achieving the minimum convergent time q ($x[q] = 0$), in general.

2.3. Problem Statement

Our problem is the following:

Problem 1 *Given an integer $N > 0$. For each $q = 1, 2, \cdots, N$, compute the maximal attractive region $\mathcal{X}_q$ satisfying for any $x_0 \in \mathcal{X}_q$ there exists an input $\boldsymbol{u}_q$ such that $z[k; x_0, \boldsymbol{u}_q] \in \mathcal{Z}$, $k = 0, 1, \cdots, q-1$ and $x[q; x_0, \boldsymbol{u}_q] = 0$, where*

$$\boldsymbol{u}_q = [u[0] \quad u[1] \quad \cdots \quad u[q-1]]^\top \in \mathbb{R}^q. \tag{5}$$

Moreover, derive a piecewise linear state feedback control law $u = f(x)$.

By computing the maximal attractive regions $\{\mathcal{X}_q\}_{q=1}^N$, we can achieve faster convergence to the origin than the ROC do for some initial state x_0.

3. Main Results

3.1. Construction of $\mathcal{X}_N$

Given $q \in \{1, 2, \cdots, N\}$ and $x[0] = x_0$. For each $k \in \{0, 1, \cdots, q\}$, the solution $x[k]$ of (1) and $z[k]$ are represented by

$$\begin{aligned} x[k] &= A^k x_0 + A^{k-1} b u[0] + \cdots + b u[k-1], \\ z[k] &= L A^k x_0 + L A^{k-1} b u[0] + \cdots + L b u[k-1] \\ &\qquad + D u[k]. \end{aligned}$$

Since we assume that A is nonsingular, the boundary condition that $x[q] = 0$ is represented by

$$x[q] = A^q (x_0 + \boldsymbol{M}_q \boldsymbol{u}_q) = 0, \tag{6}$$

where

$$\boldsymbol{M}_q = \begin{bmatrix} A^{-1} b & A^{-2} b & \cdots & A^{-q} b \end{bmatrix} \in \mathbb{R}^{n \times q}. \tag{7}$$

The constraint $Hz[k] \le h$, $k = 0, 1, \cdots, q-1$ is represented by

$$\boldsymbol{a}_q x_0 + \boldsymbol{T}_q \boldsymbol{u}_q \le \boldsymbol{h} = [h^\top \cdots h^\top]^\top \in \mathbb{R}^{q N_c} \tag{8}$$

where $\boldsymbol{a}_q = \begin{bmatrix} (HL)^\top & (HLA)^\top & \cdots & (HLA^{q-1})^\top \end{bmatrix}^\top \in \mathbb{R}^{q N_c \times n}$,

$$\boldsymbol{T}_q = \begin{bmatrix} HD & 0 & \cdots & 0 \\ HLb & HD & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ HLA^{q-2} b & HLA^{q-3} b & \cdots & HD \end{bmatrix} \in \mathbb{R}^{q N_c \times q}$$

The maximal attractive region $\mathcal{X}_q$ is characterized as follows:

$$\begin{aligned} \mathcal{X}_q = \{x_0 :\ & \boldsymbol{a}_q x_0 + \boldsymbol{T}_q \boldsymbol{u}_q \le \boldsymbol{h}, \\ & x_0 + \boldsymbol{M}_q \boldsymbol{u}_q = 0 \text{ for some } \boldsymbol{u}_q\}. \end{aligned} \tag{9}$$

Since (A, b) is a reachable pair and since A is nonsingular,

$$\operatorname{rank} \boldsymbol{M}_q = \begin{cases} q, & q \le n, \\ n, & q > n, \end{cases} \tag{10}$$

Therefore, $\boldsymbol{M}_q$ is column full rank when $q \le n$ and is full row rank when $q \ge n$, and we have the following:

Lemma 1 *Let $q < n$ and $\boldsymbol{M}_{q,L} = (\boldsymbol{M}_q^\top \boldsymbol{M}_q)^{-1} \boldsymbol{M}_q^\top$. Then, for any $x_0 \in \operatorname{Im} \boldsymbol{M}_q$, the unique solution $\boldsymbol{u}_q$ of (6) is given by*

$$\boldsymbol{u}_q = -\boldsymbol{M}_{q,L} x_0 \tag{11}$$

and $\mathcal{X}_q$ defined by (9) is given by

$$\mathcal{X}_q = \{x_0 = \boldsymbol{M}_q \xi : (\boldsymbol{a}_q - \boldsymbol{T}_q \boldsymbol{M}_{q,L}) \boldsymbol{M}_q \xi \le \boldsymbol{h}\}, \tag{12}$$

which is the largest polyhedral set such that for any $x_0 \in \mathcal{X}_q$ the input $\boldsymbol{u}_q$ defined by (11) gives that $z[k; x_0, \boldsymbol{u}_q] \in \mathcal{Z}$, $k = 0, 1, \cdots, q-1$ and $x[q; x_0, \boldsymbol{u}_q] = 0$. □

Lemma 2 *Let $q = n$. Then for any $x_0 \in \mathbb{R}^n$, the unique solution $\boldsymbol{u}_q$ of (6) is given by*

$$\boldsymbol{u}_n = -\boldsymbol{M}_n^{-1} x_0 \tag{13}$$

and $\mathcal{X}_q$ defined by (9) is given by

$$\mathcal{X}_n = \{x_0 : (\boldsymbol{a}_q - \boldsymbol{T}_q \boldsymbol{M}_q^{-1}) x_0 \le \boldsymbol{h}\}, \tag{14}$$

which is the largest polyhedral set such that for any $x_0 \in \mathcal{X}_n$ the input $\boldsymbol{u}_n$ defined by (13) gives that $z[k; x_0, \boldsymbol{u}_q] \in \mathcal{Z}$, $k = 0, 1, \cdots, n-1$ and $x[n; x_0, \boldsymbol{u}_n] = 0$. □

Lemma 3 *Let* $q > n$, $\boldsymbol{M}_{q,R} = \boldsymbol{M}_q^\top(\boldsymbol{M}_q\boldsymbol{M}_q^\top)^{-1}$, *and* $\boldsymbol{P}_q \in \mathbb{R}^{q\times(q-n)}$ *be a matrix whose column vectors are basis of* $\mathrm{Ker}\boldsymbol{M}_q$.

Then, for any $x_0 \in \mathbb{R}^n$, *any solution* $\boldsymbol{u}_q$ *of* (6) *is given by*

$$\boldsymbol{u}_q = -\boldsymbol{M}_{q,R}x_0 + \boldsymbol{P}_q\xi, \tag{15}$$

for some $\xi \in \mathbb{R}^{(q-n)}$. *Let*

$$\Upsilon_q = \{(x_0,\xi) \in \mathbb{R}^n \times \mathbb{R}^q : (\boldsymbol{a}_q - \boldsymbol{T}_q\boldsymbol{M}_{q,R})x_0 + \boldsymbol{T}_q\boldsymbol{P}_q\xi \le \boldsymbol{h}\}. \tag{16}$$

We assume that Υ_q *is bounded,* [1] *and* $\mathrm{node}\,\Upsilon_q = \{v_{q,j} = (x_{q,j},\xi_{q,j})\}_{j=1}^{N_q}$ *is the set of node of polytope* Υ_q. *Then,* $\mathcal{X}_q$ *defined by* (9) *is given by*

$$\mathcal{X}_q = \mathrm{conv}\,\{x_{q,1}, x_{q,2}, \cdots, x_{q,N_q}\}. \tag{17}$$

Let $\mathrm{node}\,\mathcal{X}_q = \{x_{q,j_1}, \cdots, x_{q,j_{m_q}}\}$ *is the set of vertices of the polytope* $\mathcal{X}_q$. *Then, for any* $x_0 \in \mathcal{X}_q$ *there exists* $\{\lambda_i \in [0,1]\}_{i=1}^{m_q}$ *such that*

$$x_0 = \sum_{i=1}^{m_q}\lambda_i x_{q,j_i}, \quad \sum_{i=1}^{m_q}\lambda_i = 1. \tag{18}$$

Define

$$\boldsymbol{u}_q = \sum_{i=1}^{m_q}\lambda_i\boldsymbol{u}_{q,j_i}, \tag{19}$$

$$\boldsymbol{u}_{q,j_i} = -\boldsymbol{M}_{q,R}x_{q,j_i} + \boldsymbol{P}_q\xi_{q,j_i}, \tag{20}$$

where $v_{q,j} = (x_{q,j},\xi_{q,j}) \in \mathrm{node}\,\Upsilon_q$.

The polytope $\mathcal{X}_q$ *is the largest polytope such that for any* $x_0 \in \mathcal{X}_q$ *the input* $\boldsymbol{u}_q$ *defined by* (19) *gives that* $z[k; x_0, \boldsymbol{u}_q] \in \mathcal{Z}$, $k = 0,1,\cdots,q-1$ *and* $x[q; x_0, \boldsymbol{u}_q] = 0$. □

We summarize above observations and get the following:

Theorem 1 *Assume that* $N \ge n$ *and* $\{\mathcal{X}_q\}_{q=1}^{n-1}$ *are defined by* (12) *and* $\mathcal{X}_n$ *is defined by* (14). *Moreover, we assume that* $\{\Upsilon_q\}_{q=n+1}^N$ *defined by* (16) *are bounded and* $\{\mathcal{X}_q\}_{q=n+1}^N$ *are defined by* (17).

Then, we have the following;

(a) *The polytope* $\mathcal{X}_q$ *is the largest polytope such that for any* $x_0 \in \mathcal{X}_q$ *there exists a* $\boldsymbol{u}_q$ *such that* $z[k; x_0, \boldsymbol{u}_q] \in \mathcal{Z}$, $k = 0,1,\cdots,q-1$ *and* $x[q; x_0, \boldsymbol{u}_q] = 0$.

(b) *If* $x_0 \in \mathcal{X}_q$ *for some* $q < N-1$, *then* $x_0 \in \mathcal{X}_{q+1}$. *Therefore, we have*

$$\mathcal{X}_1 \subseteq \cdots \subseteq \mathcal{X}_q \subseteq \cdots \subseteq \mathcal{X}_N. \tag{21}$$

(c) *If* $x_0 \in \mathcal{X}_q$ *for some* $q > 1$, *then* $x[1] \in \mathcal{X}_{q-1}$. □

[1] A sufficient condition that Υ_q is bounded is all elements of x and u are elements of z.

For the system given in **Example 1**, we computed $\{\mathcal{X}_q\}_{q=1}^N$ where $N = 5$. The gray region in **Figure 1** is $\mathcal{X}_5$, which is much larger than attractive regions $\mathcal{X}_5(x_0)$ computed by applying ROC.

Suppose that we computed polytopes $\{\mathcal{X}_q\}_{q=1}^N$ under the assumptions in **Theorem 1**. Let

$$n_q = \dim\mathcal{X}_q. \tag{22}$$

Note that $n_q \le q$ and $n_q < m_q$ for all $q = 1,2,\cdots,N$, where m_q is the cardinality of the nodes of $\mathcal{X}_q$. Since $h > 0$ by the assumption, $\mathcal{X}_p$ includes $0 \in \mathbb{R}^n$ as an interior point, and, hence, $n_q = n$ for all $q = n+1, n+2, \cdots, N$.

3.2. Control law

In this subsection, we will state how to determine the control law $u = f(x)$. If $q \le n$, then $\boldsymbol{u}_q$ is defined uniquely. On the other hand, when $q > n$, $\boldsymbol{u}_q$ depends on the choice of $\{\lambda_i \in [0,1]\}_{i=1}^{m_q}$ satisfying (18), which is not unique in general. Let $\Delta_q = \mathcal{X}_q \backslash \mathcal{X}_{q-1}$. [2] We divide Δ_q into simplexes $\{\mathcal{S}_{q,\ell}\}_{\ell=1}^{d_q}$, that is,

$$\Delta_q = \bigcup_{\ell=1}^{d_q}\mathcal{S}_{q,\ell}, \quad \mathrm{int}\,\mathcal{S}_{q,\ell} \cap \mathrm{int}\,\mathcal{S}_{q,\ell'} = \emptyset,\ \ell \ne \ell' \tag{23}$$

$$\mathcal{X}_q = \mathcal{X}_{q-1} \cup \left(\bigcup_{\ell=1}^{d_q}\mathcal{S}_{q,\ell}\right). \tag{24}$$

Let $\mathrm{node}\,\mathcal{S}_{q,\ell} = \{x_{q,j_{\ell,i}}\}_{i=1}^{n+1}$ be the set of nodes of the simplex $\mathcal{S}_{q,\ell}$, where $(x_{q,j_{\ell,i}}, \xi_{q,j_{\ell,i}}) \in (\mathrm{node}\,\mathcal{X}_q \cup \mathrm{node}\,\mathcal{X}_{q-1})$. Then, for each $x_0 \in \mathcal{S}q,\ell$, there is a unique $\lambda \in \mathbb{R}^{n+1}$ such that $x_0 = \tilde{X}_{q,\ell}\lambda$, where

$$\tilde{X}_{q,\ell} = \begin{bmatrix} x_{q,j_{\ell,1}} & x_{q,j_{\ell,2}} & \cdots & x_{q,j_{\ell,n+1}} \\ 1 & 1 & \cdots & 1 \end{bmatrix}. \tag{25}$$

Note that $\tilde{X}_{q,\ell} \in \mathbb{R}^{(n+1)\times(n+1)}$ is nonsingular since $\mathcal{S}_{q,\ell}$ is a simplex. Therefore $\lambda = \tilde{X}_{q,\ell}^{-1}x_0$. We define $\boldsymbol{u}_q$ by

$$\boldsymbol{u}_q = U_{q,\ell}\tilde{X}_{q,\ell}^{-1}x_0, \tag{26}$$

where

$$U_{q,\ell} = \begin{bmatrix}\boldsymbol{u}_{q,j_{\ell,1}} & \boldsymbol{u}_{q,j_{\ell,2}} & \cdots & \boldsymbol{u}_{q,j_{\ell,n+1}}\end{bmatrix} \tag{27}$$

$$\boldsymbol{u}_{q,j_{\ell,i}} = -\boldsymbol{M}_{q,R}x_{q,j_{\ell,i}} + \boldsymbol{P}_q\xi_{q,j_{\ell,i}}. \tag{28}$$

We summarize as follows:

Theorem 2 *Assume that the assumptions of* **Theorem 1** *is satisfied. Then, the polytope* $\mathcal{X}_q$ *is the largest polytope such that for any* $x_0 \in \mathcal{X}_q$ *there exists a* $\boldsymbol{u}_q$ *such that* $z[k; x_0, \boldsymbol{u}_q] \in \mathcal{Z}$, $k = 0,1,\cdots,q-1$ *and* $x[q; x_0, \boldsymbol{u}_q] = 0$, *and input* $\boldsymbol{u}_q$ *is given by* (11) *when* $q < n$, (13) *when* $q = n$, *and* (26) *when* $q > n$. □

[2] We note here that $\mathcal{X}_{q-1} \subseteq \mathcal{X}_q$ because of **Theorem 1**, (b)

For the system given in **Example 1**, we computed $\{\mathcal{X}_q\}_{q=1}^n$ and $\{\mathcal{S}_{q,\ell}\}$ for $q = n+1, n+2, \cdots, N$, where $n = 2$ and $N = 5$. In **Figure 2**, $\mathcal{X}_1$ is a red line segment, $\mathcal{X}_2$ is the yellow polytope, $\{\mathcal{S}_{3,\ell}\}$ are green simplexes, $\{\mathcal{X}_{4,\ell}\}$ are cyan simplexes, and $\{\mathcal{S}_{5,\ell}\}$ are magenta simplexes. We also show $\{\mathcal{X}_q(x_0)\}_{q=1}^5$ computed by ROC in **Figure 3**, where $x_0 = [-2\ 5]^\top$. In **Figure 3**, $\mathcal{X}_1(x_0)$ is a red line segment, $\mathcal{X}_2(x_0)$ is a yellow polytope, $\Delta_3(x_0) = \mathcal{X}_3(x_0)\backslash\mathcal{X}_2(x_0)$ is divided into green simplexes, $\Delta_4(x_0) = \mathcal{X}_4(x_0)\backslash\mathcal{X}_3(x_0)$ is divided into cyan simplexes, and $\Delta_5(x_0) = \mathcal{X}_5(x_0)\backslash\mathcal{X}_4(x_0)$ is divided into magenta simplexes. We can see that $\mathcal{X}_5$ obtained our method is larger than $\mathcal{X}_5(x_0)$. This is the first contribution of our method.

Let $x_0 = [-2\ 4]$. Note that $x_0 \in \Delta_4$ in **Figure 2**, and, hence, the trajectory $x[k; x_0, \boldsymbol{u}_4]$ obtained by our method converges 0 by 4 steps. On the other hand, $x_0 \in \Delta_5([-2\ 5]^\top)$ in **Figure 3**, and, hence, the trajectory $x[k; x_0, \boldsymbol{u}_5]$ obtained by ROC need 5 steps converges 0. Thus, the trajectory by our method converges 0 by smaller steps than that of ROC. This is the second contribution of our method.

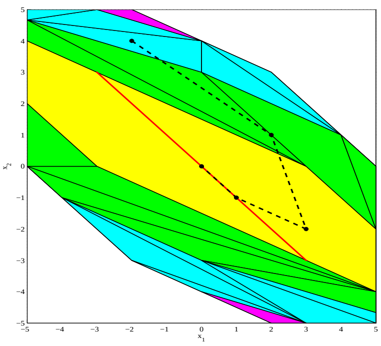


Figure 2: The partitioned state space and state trajectory from point $x[0] = [-2\ 4]^\top$

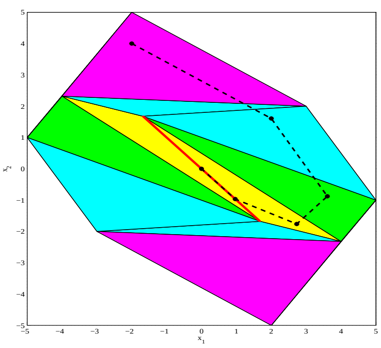


Figure 3: The partitioned state space and state trajectory from point $x[0] = [-2\ 4]^\top$ by ROC method

4. Concluding Remark

In this paper, we assume that exact model of the plant is given and there is no disturbance and noises. However, in practice, we can not expect such a situation. Suppose that $N > n$ and that $x_0 \in \mathcal{S}_{N,\ell}$ for some ℓ. According to the proposing scheme, we have a input $\boldsymbol{u}_N = \begin{bmatrix} u[0] & u[1] & \cdots & u[N-1] \end{bmatrix}$. Suppose that, like the model predictive control scheme, we only apply $u[0]$ to the system, and observe the state $x_0' = x[1]$ at time 1. Note that $x_0^{[1]}$ might be different from $x[1; x_0, \boldsymbol{u}_N]$ because of model errors, disturbances or noises. But it is not so ambiguous that we expect that $x_0^{[1]} \in \mathcal{X}_{N-1}$ because of **Theorem 1**, (c). Then, determine ℓ' such that $x_0^{[1]} \in \mathcal{S}_{N-1,\ell'}$, compute $\boldsymbol{u}_{N-1} = \begin{bmatrix} u'[0] & u'[1] & \cdots & u'[N-2] \end{bmatrix}$, and apply only $u'[0]$ to the system. By repeating this process $N-n$ times, we can expect that $x_0^{[n]} = x[N-n] \in \mathcal{X}_n$. However, since $\dim \mathcal{X}_q < n$ for $q < n$, it is not reasonable to expect that $x[N] = 0$ when there are model errors, disturbances or noises. This is the point we need to circumvent in future study.

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