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Derivation Method of the Bifurcation Point for the Periodic Solution in an Impact Oscillator with Periodic Local Cross-Section

Goki Ikeda†, Hiroyuki Asahara†, Kazuyuki Aihara‡ and Takuji Kousaka†

†Faculty of Engineering, Oita University
700 Dannoharu, Oita, Oita, 870-1192 Japan
‡Institute of Industrial Science, The University of Tokyo,
7-3-1, Hongo, Bukyo-ku, Tokyo, Japan
Email: goki@bifurcation.jp, aihara@sat.t.u-tokyo.ac.jp, takuji@oita-u.ac.jp

Abstract—This paper presents a derivation method of the bifurcation point for the periodic solution in an impact oscillator with periodic local cross-section. First, we explain the impact model and construct the Poincaré map. The construction of the Poincaré map has been subjected by considering presence or absence of the impact. Next, we show the Jacobian matrix and specify the derivative of the Poincaré map to calculate the bifurcation point. Finally, the proposed method is applied for an impact oscillator with periodic local cross-section.

1. Introduction

The bifurcation analysis and the stability analysis are important for understanding the qualitative property of the nonlinear dynamic systems. Therefore, many researchers have been analyzed the smooth system since old times [1]. In particular, AUTO and BunKi are useful tools for tracking a bifurcation point in these systems [2, 3]. However, these tools may seem difficult to apply to the switching system depending on its state or a periodic interval which has the interrupted characteristics.

On the other hand, versatile tools for the switching system have not developed. Also, the impact oscillator, which is one of the switching system is received attention in recent years. The impact oscillator has the characteristics that the solution jumps when the trajectory hits the border. Above all, in particular, many systems with moving obstacle are studied in mechanical system. For instance, gear ratting [4, 5], rotor-casing dynamical system [6], impact damper [7], and so on. In order to analyze these systems, Yoshitake proposed the shooting method [8] which can analyze the stability of the periodic solution. However, there are no method to calculate the bifurcation point directly. Therefore, for the purpose of the development of the bifurcation analysis, we have proposed a calculation method of bifurcation point for an impact oscillator with periodic function in previous work [9]. Here, various periodic solutions are observed in the impact oscillators with moving obstacle, but the method in Ref. [9] is only effective against period-1 solution. Hence, we present a derivation method of the bifurcation point for the periodic solution in an impact oscillator with periodic local cross-section in this paper. The proposed method can calculate the bifurcation point for not only period-1 solution but also periodic solution.

First of all, we show the physical model and explain its dynamics. Furthermore, the Poincaré map is constructed by considering presence or absence of the impact. Next, the Jacobian matrix and its elements are described. Then, we specify the derivative of the Poincaré map to calculate the bifurcation point of the periodic solution. Finally, we apply this method for an impact oscillator with periodic local cross-section to confirm the validity of the method.

2. An impact oscillator with periodic local cross-section

We apply the method to an impact oscillator with periodic local cross-section shown in Fig. 1. This system is modelled by using a spring, damper and mass, respectively and is equivalent to the rigid overhead wire-pantograph model [10]. Also, the overhead wire model represents the rail corrugation, which is described as the periodic vibration. Note that the mass of the pantograph model impacts the overhead wire model. The considered model can be described by the following differential equations

$$\begin{cases} \dfrac{dx}{dt} = v \\ \dfrac{dv}{dt} = -x - 2\zeta v \end{cases}, \tag{1}$$

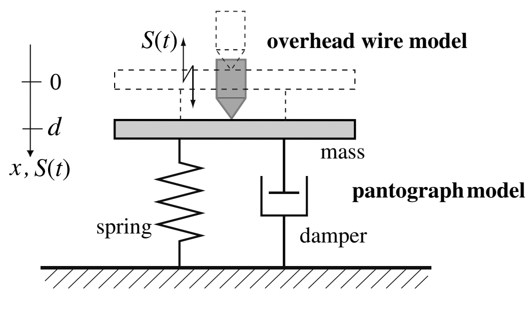

Figure 1: A rigid overhead wire-pantograph system.

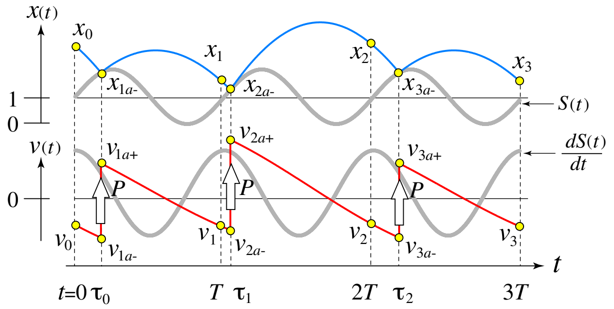

Figure 2: Example of the Period-2 solution.

where ζ, x, and v in the pantograph model are expressed in a damping ratio, the displacement, and the velocity. The normalized equation of the overhead wire model is given by

$$S(t) = \varepsilon \sin \Omega t + 1, \tag{2}$$

where $S(t)$, ε, and Ω are expressed in the displacement of rigid overhead wire, the amplitude, and the angular frequency here. We show the example of the Period-2 solution in Fig. 2. When $x(t)$ reaches to $S(t)$, the velocity of mass changes as

$$v_+ = -\alpha v_- + (1+\alpha)\frac{dS(t)}{dt}. \tag{3}$$

Note that v_+ is the velocity after the impact, and v_- is the previous velocity. Also, α is a coefficient of restitution between the pantograph model and the overhead wire model.

3. Analytical method and result

3.1. Poincaré map

We calculate the bifurcation point in the proposed system. First of all, the solutions can be described by the following two-dimensional system

$$\begin{cases} \dfrac{dx}{dt} = f(x, v, \boldsymbol{\lambda}) \\ \dfrac{dv}{dt} = g(x, v, \boldsymbol{\lambda}) \end{cases}, \tag{4}$$

where the parameters t, x, v and λ satisfy $T \in \boldsymbol{R}$, $x, v \in \boldsymbol{R}^2$, $f, g : \boldsymbol{R}^2 \to \boldsymbol{R}^2$. Now, Eq. (4) is written as

$$\begin{cases} x(t) = \varphi(t; x_0, v_0, \boldsymbol{\lambda}), \ x(0) = x_0 \\ v(t) = \phi(t; x_0, v_0, \boldsymbol{\lambda}), \ v(0) = v_0 \end{cases} \tag{5}$$

where x_0 and v_0 means the initial value at time $t = 0$. Next, we define the following local section $\Pi \in \boldsymbol{R}^2$ by using scalar function q : $\boldsymbol{R}^2 \in \boldsymbol{R}^2$.

$$\Pi = \{x, v \in \boldsymbol{R}^2 : q(x, v) = 0, q : \boldsymbol{R}^2 \to \boldsymbol{R}^2\}, \tag{6}$$

$$q(t+T; x, v) = q(t; x, v). \tag{7}$$

Furthermore, the maps can be categorized by the presence or absence of the impact of trajectory in $kT \le t < (k+1)T$. First, when the trajectories are smooth in this range, the map $M_k^{[1]}$ can be described as

$$\begin{aligned} M_k^{[1]} : \boldsymbol{R}^2 &\to \boldsymbol{R}^2, \\ \boldsymbol{x}_k &\mapsto \boldsymbol{x}_{k+1} = (\varphi(T; x_k, v_k, \boldsymbol{\lambda}), \phi(T; x_k, v_k, \boldsymbol{\lambda}))^\top. \end{aligned} \tag{8}$$

Next, we consider the case whose trajectories include the impact. If x reaches to Π, the velocity v is jumped by the map P. The map P is written as follows:

$$\begin{aligned} P : \boldsymbol{R}^2 &\to \boldsymbol{R}^2, \\ \boldsymbol{x}_{ka-} &= (\varphi(\tau_k; x_k, v_k, \boldsymbol{\lambda}), \phi(\tau_k; x_k, v_k, \boldsymbol{\lambda}))^\top \\ &\mapsto \boldsymbol{x}_{ka+} = \left(x_{ka-}, -\alpha v_{ka-} + (1+\alpha)\frac{dS(t)}{dt}\right)^\top \end{aligned} \tag{9}$$

where τ_k denotes the time when the solution reaches to Π. The discretized solutions $\boldsymbol{x}_k$ are expressed as

$$\boldsymbol{x}_k = (\varphi(T - \tau_k; x_{ka-}, v_{ka+}, \boldsymbol{\lambda}), \phi(T - \tau_k; x_{ka-}, v_{ka+}, \boldsymbol{\lambda}))^\top. \tag{10}$$

Moreover, we define the maps around the time of the impact as follows:

$$\begin{aligned} M_{kA} &: \boldsymbol{R}^2 \to \Pi, \quad \boldsymbol{x}_k \mapsto \boldsymbol{x}_{ka-} \\ M_{kB} &: \Pi \to \boldsymbol{R}^2, \quad \boldsymbol{x}_{ka+} \mapsto \boldsymbol{x}_{k+1}. \end{aligned} \tag{11}$$

Consequently, the map $M_k^{[2]}$ is given by

$$\begin{aligned} &M_k^{[2]} : \boldsymbol{R}^2 \to \boldsymbol{R}^2 \\ &\quad \boldsymbol{x}_k \mapsto \boldsymbol{x}_{k+1} = M_{kB} \circ P \circ M_{kA}. \end{aligned} \tag{12}$$

Additionally, the Poincaré map of period-m can be expressed by combining eq.(9) and eq.(12).

$$\begin{aligned} &M : \boldsymbol{R}^2 \to \boldsymbol{R}^2 \\ &\quad \boldsymbol{x}_0 \mapsto \boldsymbol{x}_m = M_{m-1}^{[i]} \circ \cdots \circ M_1^{[i]} \circ M_0^{[i]}, \\ &\qquad\qquad i = 1 \text{ or } 2. \end{aligned} \tag{13}$$

3.2. Jacobian matrix

A fixed point of the Poincaré map is given by

$$\boldsymbol{x}_0 - M(\boldsymbol{x}_0) = \begin{bmatrix} x_0 - [1\ 0]M(x_0, v_0) \\ v_0 - [0\ 1]M(x_0, v_0) \end{bmatrix} = \boldsymbol{0}. \tag{14}$$

The characteristic equation for the fixed point is expressed

$$\chi(\mu) = \det|\mu \boldsymbol{I}_2 - DM(\boldsymbol{x}_0)|. \tag{15}$$

Therefore, simultaneous equation of a fixed point and the characteristic equation is written as

$$F(x_0, v_0, \lambda_a) = \begin{bmatrix} x_0 - [1\ 0]M(x_0, v_0) \\ v_0 - [0\ 1]M(x_0, v_0) \\ \chi(\mu) \end{bmatrix} = \boldsymbol{0}. \tag{16}$$

Here, we decide the characteristic multiplier by the type of bifurcation. For example, if we calculate the bifurcation point of the period-doubling bifurcation, the characteristic multiplier is fixed as -1.0. On the other hand, in case of the saddle-node bifurcation, we fix the characteristic multiplier as 1.0. Also, Eq. (16) can be calculated for the unknown variables, x_0, v_0, and a bifurcation parameter λ by using Newton's method. Then, the Jacobian matrix of F is

$$DF(x_0, v_0, \lambda) = \begin{bmatrix} 1 - \dfrac{\partial M}{\partial x_0} & -\dfrac{\partial M}{\partial v_0} & -\dfrac{\partial M}{\partial \lambda} \\ -\dfrac{\partial M}{\partial x_0} & 1 - \dfrac{\partial M}{\partial v_0} & -\dfrac{\partial M}{\partial \lambda} \\ \dfrac{\partial \chi(\mu)}{\partial x_0} & \dfrac{\partial \chi(\mu)}{\partial v_0} & \dfrac{\partial \chi(\mu)}{\partial \lambda} \end{bmatrix}. \quad (17)$$

Here, we discuss the elements of eq.(17). The derivative of the Poincaré map depending on the initial value is expressed as

$$DM(\boldsymbol{x}_0) = \frac{\partial M}{\partial \boldsymbol{x}_0} = \prod_{k=1}^{m} \frac{\partial M_{m-k}}{\partial \boldsymbol{x}_{m-k}} \quad (18)$$

where the $\partial M_k/\partial x_k$ can be grouped by the presence or absence of the impact of trajectory per period. In case of no impact per period, $\partial M_k/\partial x_k$ is written as follows:

$$\begin{bmatrix} \dfrac{\partial [1\ 0] M_k^{[1]}}{\partial x_k} & \dfrac{\partial [1\ 0] M_k^{[1]}}{\partial v_k} \\ \dfrac{\partial [0\ 1] M_k^{[1]}}{\partial x_k} & \dfrac{\partial [0\ 1] M_k^{[1]}}{\partial v_k} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \varphi}{\partial x_k} & \dfrac{\partial \varphi}{\partial v_k} \\ \dfrac{\partial \phi}{\partial x_k} & \dfrac{\partial \phi}{\partial v_k} \end{bmatrix}. \quad (19)$$

On the other hand, when the trajectories have the impact, $\partial M_k/\partial x_k$ is shown as

$$\begin{aligned} &\begin{bmatrix} \dfrac{\partial [1\ 0] M_k^{[2]}}{\partial x_k} & \dfrac{\partial [1\ 0] M_k^{[2]}}{\partial v_k} \\ \dfrac{\partial [0\ 1] M_k^{[2]}}{\partial x_k} & \dfrac{\partial [0\ 1] M_k^{[2]}}{\partial v_k} \end{bmatrix} \\ &= \begin{bmatrix} \left.\dfrac{\partial \varphi}{\partial t}\right|_{t=T-\tau_k} \\ \left.\dfrac{\partial \phi}{\partial t}\right|_{t=T-\tau_k} \end{bmatrix} \begin{bmatrix} -\dfrac{\partial \tau_k}{\partial x_k} & -\dfrac{\partial \tau_k}{\partial v_k} \end{bmatrix} + \begin{bmatrix} \dfrac{\partial \varphi}{\partial x_{ka+}} & \dfrac{\partial \varphi}{\partial v_{ka+}} \\ \dfrac{\partial \phi}{\partial x_{ka+}} & \dfrac{\partial \phi}{\partial v_{ka+}} \end{bmatrix} \\ &\quad \left\{ \begin{bmatrix} 0 \\ \dfrac{\partial}{\partial t}\left((1+\alpha)\dfrac{dS(\tau_k)}{dt}\right) \end{bmatrix} \begin{bmatrix} \dfrac{\partial \tau_k}{\partial x_k} & \dfrac{\partial \tau_k}{\partial v_k} \end{bmatrix} \right. \\ &\quad \left. + \begin{bmatrix} 1 & 0 \\ 0 & -\alpha \end{bmatrix} \begin{bmatrix} \dfrac{\partial x_{ka-}}{\partial x_k} & \dfrac{\partial x_{ka-}}{\partial v_k} \\ \dfrac{\partial v_{ka-}}{\partial x_k} & \dfrac{\partial v_{ka-}}{\partial v_k} \end{bmatrix} \right\}. \end{aligned} \quad (20)$$

Furthermore,

$$\begin{aligned} \begin{bmatrix} \dfrac{\partial x_{ka-}}{\partial x_k} & \dfrac{\partial x_{ka-}}{\partial v_k} \\ \dfrac{\partial v_{ka-}}{\partial x_k} & \dfrac{\partial v_{ka-}}{\partial v_k} \end{bmatrix} &= \begin{bmatrix} \left.\dfrac{\partial \varphi}{\partial t}\right|_{t=T-\tau_k} \\ \left.\dfrac{\partial \phi}{\partial t}\right|_{t=T-\tau_k} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \tau_k}{\partial x_k} & \dfrac{\partial \tau_k}{\partial v_k} \end{bmatrix} \\ &\quad + \begin{bmatrix} \dfrac{\partial \varphi}{\partial x_k} & \dfrac{\partial \varphi}{\partial v_k} \\ \dfrac{\partial \phi}{\partial x_k} & \dfrac{\partial \phi}{\partial v_k} \end{bmatrix}. \end{aligned} \quad (21)$$

We should remark that the function

$$q(\tau_k(x_k, v_k); x_k, v_k, \boldsymbol{\lambda}) = 0 \quad (22)$$

is differentiable for $\boldsymbol{x}_k$. Hence, $\partial \tau_k/\partial x_k$ and $\partial \tau_k/\partial v_k$ can be obtained. Also, we can differentiate other elements in much the same way.

3.3. Application result

We show the application result of our method in Table 1 which represents the bifurcation points for various of Ω and ζ. PD^m in the table indicate the period-doubling bifurcation for the period-m solutions. The bifurcation point can be obtained by solving Eq.(17). Next, Table 2 shows the result in the stability analysis of the periodic solution with period-1 and period-2. This table can be calculated by solving Eq. (14), (15). Also, Fig. 3 is the one-parameter bifurcation diagram upon varying the bifurcation parameter Ω from $\Omega = 6.0$ to $\Omega = 6.45$. Each values of Ω labeled by (a), (b), and (c) in Fig. 3 correspond to Fig. 4 which indicates the phase planes. The points in the phase planes are the periodic solution. Period-1 solution, period-2 solution, and period-4 solution are observed before or after the bifurcation point. If $\zeta = 0.4$, then the value of Ω in Table 1 can be obtained in correspondence to Table 2 and Fig. 3.

4. Conclusion

In this paper, we have proposed a derivation method of the bifurcation point for the periodic solution in an impact oscillator with periodic local cross-section. First, we defined the Poincaré map to express the fixed point and the characteristic equation. Next, we showed the Jacobian matrix and the derivative of the Poincaré map in order to calculate the bifurcation point. Finally, we applied this method for an impact oscillator with periodic local cross-section, and we could confirm the validity of this method. The future work is establishment of the method response to the impact oscillator.

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Table 1: Calculation of the bifurcation point (α = 0.5, ε = 0.068).

	Period-1				Period-2			
ζ	Ω	μ_1	μ_2	Remarks	Ω	μ_1	μ_2	Remarks
0.2000	5.8216	-0.1613	-1.0000	PD[1]	6.0818	-0.0273	-1.0000	PD[2]
0.3000	5.9871	-0.1331	-1.0000	PD[1]	6.2534	-0.0187	-1.0000	PD[2]
0.4000	6.1459	-0.1103	-1.0000	PD[1]	6.4185	-0.0130	-1.0000	PD[2]
0.5000	6.2983	-0.0922	-1.0000	PD[1]	6.5773	-0.0092	-1.0000	PD[2]
0.6000	6.4446	-0.0776	-1.0000	PD[1]	6.7299	-0.0066	-1.0000	PD[2]

Table 2: Calculation of the characteristic multiplier (α = 0.5, ε = 0.068, ζ = 0.4).

Period-1				Period-2			
Ω	μ_1	μ_2	Remarks	Ω	μ_1	μ_2	Remarks
6.1350	-0.1123	-0.9803	Stable	6.4050	-0.0145	-0.8923	Stable
6.1400	-0.1114	-0.9893	Stable	6.4100	-0.0139	-0.9318	Stable
6.1450	-0.1105	-0.9983	Stable	6.4150	-0.0132	-0.9715	Stable
6.1459	-0.1103	-1.0000	PD[1]	6.4185	-0.0130	-1.0000	PD[2]
6.1500	-0.1096	-1.0072	Unstable	6.4200	-0.0129	-1.0112	Unstable

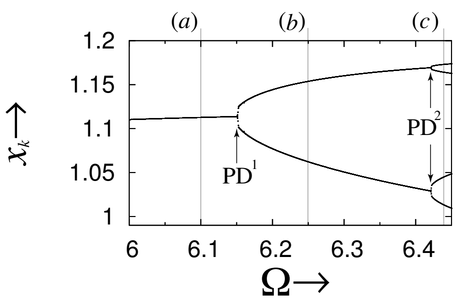
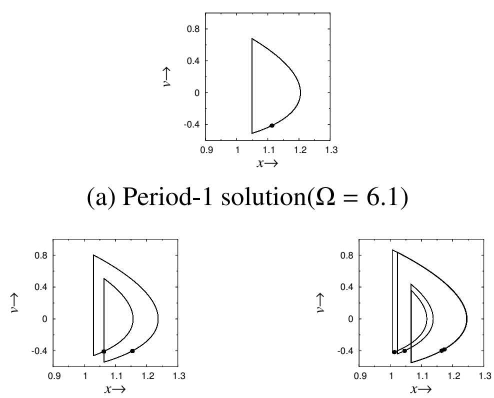


Figure 3: One-parameter bifurcation diagram (α = 0.5, ε = 0.068, ζ = 0.4).



(a) Period-1 solution(Ω = 6.1)

(b) Period-2 solution (Ω = 6.25)

(c) Period-4 solution (Ω = 6.44)

Figure 4: Phase planes (α = 0.5, ε = 0.068, ζ = 0.4).

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