IEICE The Institute of Electronics, Information and Communication Engineers

IEICE Proceeding Series

Chaos and Oscillation Death in a Weakly Driven BVP Oscillator

Yoshimasa Shinotsuka, Naohiko Inaba, Munehisa Sekikawa, Tetsuro Endo

Vol. 1 pp. 887-890
Publication Date: 2014/03/17
Online ISSN: 2188-5079

2012 International Symposium on Nonlinear Theory and its Applications
NOLTA2012, Palma, Majorca, Spain, October 22-26, 2012

Chaos and Oscillation Death in a Weakly Driven BVP Oscillator

Yoshimasa Shinotsuka†, Naohiko Inaba‡, Munehisa Sekikawa*, and Tetsuro Endo†

†Department of Electronics and Bioinformatics, Meiji University, 214-8571 Kawasaki, Japan
‡Organisation for the Strategic Coordination of Research and Intellectual Property,
Meiji University, 214-8571 Kawasaki, Japan
*Institute of Industrial Science, the University of Tokyo, 153-8505 Tokyo, Japan
Email: ce21043@meiji.ac.jp, naohiko@yomogi.jp

Abstract—We discuss the bifurcation structure of oscillation death and chaos generated in a weakly driven BVP oscillator where the parameter values are chosen such that a stable focus and a stable relaxation oscillation coexist in close proximity when no perturbation is applied. Chaos and oscillation death coexist in the weakly driven BVP oscillator at $B_1 = 0.002$ where B_1 is an amplitude of the forcing term. A sudden disappearance of a chaotic oscillation is confirmed as B_1 increases, and only oscillation death becomes an attractor. The basin of oscillation death is derived numerically. The boundaries of the basin are extremely complex. When we increase B_1 a little, it is found that the basin becomes extremely large, and the solutions for all initial conditions converge to oscillation death at $B_1 = 0.004$. It means that chaos drastically disappears when the amplitude of the forcing term is extremely weak.

1. Introduction

The autonomous Bonhoeffer-van der Pol (BVP) circuit is a fundamental oscillator that generates a limit cycle. It has an additional linear resistor in series with the inductor, compared with the van der Pol oscillator. This difference seems to be marginal at first glance. However, there is a noteworthy difference in these two oscillators. It is pointed out by Rabinovitch and Rogachevskii.[1] that a subcritical Andronov-Hopf bifurcation (AHB) can occur when the linear resistance is chosen to be larger, whereas only a supercritical AHB is possible in the van der Pol oscillator.

How is the BVP dynamics influenced by a weak periodic perturbation? Notably complex dynamical structures are expected to emerge because the stable focus and the stable relaxation oscillation coexist in close proximity, and the solution may alternate between the focus and the relaxation oscillation under weak periodic perturbation. Actually, it generates a rich variety of interesting phenomena such as chaos with unusually complicated waveforms and complex mixed-mode oscillations[2, 3]. Rabinovitch *et al.* has not paid attention to such a situation. Therefore, the analysis of such a simple dynamics that exhibits complex bifurcations has just begun[2, 3].

Sekikawa *et al.* [3] have analyzed this oscillator, and have found a sudden change from chaos to oscillation death as a parameter is varied. Strictly speaking, "oscillation death" referred here is not a stable equilibrium but an extremely weak oscillation because the dynamics is nonautonomous and has a forcing term. The stable relaxation oscillation, which exists when no perturbation is applied, disappears subject to extremely weak periodic perturbation. However, this situation has not yet been analyzed in [3].

In this paper, we will pay attention to "oscillation death in non-autonomous systems" Sekikawa *et al.* discovered[3] in the following weakly driven BVP oscillator:

$$\begin{cases} \varepsilon\dot{x} &= y - (-x + x^3) \\ \dot{y} &= -x - k_1 y + B_0 + B_1 \sin\omega\tau, \end{cases} \tag{1}$$

where ε and B_1 are assumed to be small. The oscillator has been studied intensively because the driven BVP oscillator is known as a simplified Hodgkin-Huxley model[4, 5, 6, 7, 8]. It must be noted that Eq. (1) is an extremely simple equation that generates chaos and related bifurcation phenomena because it consists of a natural bistability generated by a subcritical AHB and an extremely weak periodic forcing term. In this equation, chaos is observed when the amplitude of the forcing term B_1 is extremely weak such as $B_1 = 0.002$. The generation of chaos is confirmed by calculating the largest Lyapunov exponent. Such chaos that appears under extremely weak periodic perturbation is not rare in slow-fast systems[3]. Slow-fast systems refer to a dynamics where one of the state variables x moves faster than the other state variable(s) except in the neighborhood of the nullcline $\dot{x} = 0$. The step size $h = 2\pi/(\omega/1024)$ is sufficient small because the reasonable bifurcation sets have been derived using Runge-Kutta solver[3].

At $B_1 = 0.002$ and $\omega = 1.35$, at least, chaos and oscillation death coexist. Sekikawa *et al.* succeeded to explain using bifurcation theory that oscillation death disappears for large B_1. This phenomenon can be understandable intuitively. However, Sekikawa *et al.* do not at all analyze the birth of oscillation death. We numerically investigate the set of initial conditions whose attractors become oscillation death. The set of initial conditions is very thin at $B_1 = 0.002$. The boundary of the set is extremely complex. As the B_1 is increased until $B_1 = 0.004$, however, all solutions at any initial conditions converge to oscillation death according to our numerical results. Chaos that exist at $B_1 = 0.003$ is merged into oscillation death, and is missing at $B_1 = 0.004$.

2. Circuit Setup of the BVP oscillator and a subcritical Andronov-Hopf bifurcation

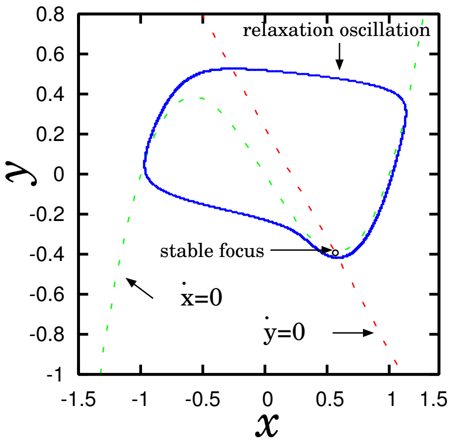


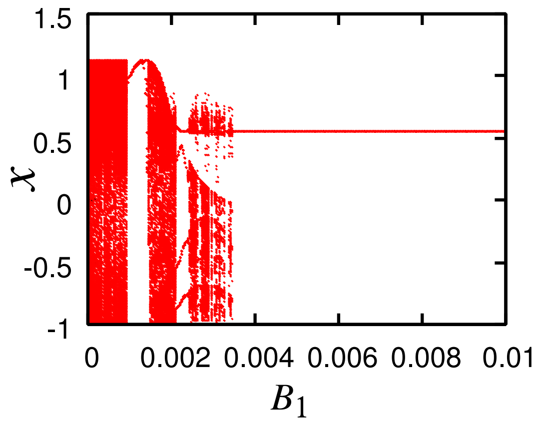
Figure 1: Two stable attractors when no perturbation is applied ($k_1 = 0.9$ and $B_0 = 0.21$).

First, we consider the case where no perturbation is applied to the circuit, namely, $B_1 = 0$. As mentioned in the Introduction, the BVP oscillator contains a linear resistor in series with the inductor, compared with the van der Pol oscillator. The existence of the linear resistance seems to be insignificant and marginal at first glance. However, a crucial bifurcation occur in the autonomous BVP oscillator due to the presence of this resistor. Rabinovitch and Rogachevskii pointed out that the autonomous BVP oscillator has a richer dynamics compared with the van der pol oscillator. They discovered that a subcritical Andronov Hopf bifurcation can take place in the presence of the linear resistor, whereas only a supercritical AHB is possible in the van der Pol oscillator[1].

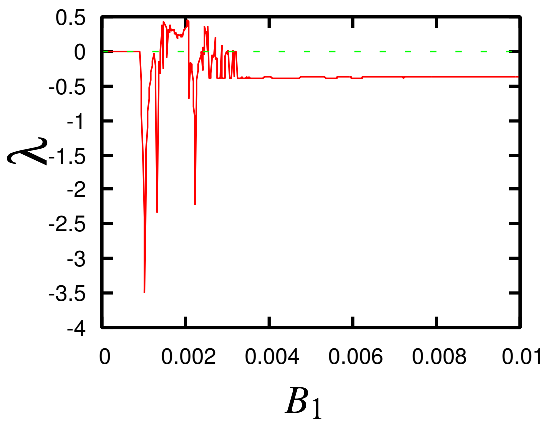
Figure 1 shows the two attractors when $B_1 = 0$. One is a stable relaxation oscillation and the other is a stable focus. The stable focus is plotted and the relaxation oscillation is drawn with a solid curve. The nullclines are also drawn with dashed curves. The interesting bifurcation occur due to the presence of larger k_1.

3. Analysis of the BVP Oscillator under weak periodic perturbation

Now we ask how the circuit dynamics is influenced by weak periodic perturbation because a stable focus and a stable relaxation oscillation coexist when no perturbation is applied. In such a system, if weak periodic perturbation is applied, complex behaviors are expected to emerge because the stable focus and the stable relaxation oscillation coexist in close proximity to each other. If the weak periodic perturbation is applied, the solution will alternate between the stable focus and the relaxation oscillation by



(a) One-parameter bifurcation diagram.



(b) Graph of the largest Lyapunov exponent.

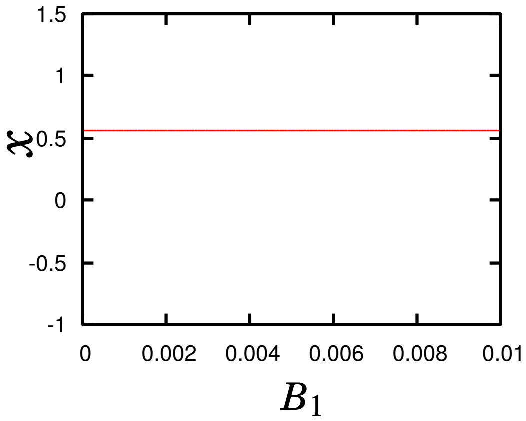
Figure 2: One-parameter bifurcation diagram and the corresponding graph of the largest Lyapunov exponent where B_1 is varied. The initial condition is $(x_0, y_0) = (3, 3)$ ($k_1 = 0.9$, $B_0 = 0.21$ and $\omega = 1.35$).

the weak periodic perturbation. Actually, the generation of complex bifurcations such as complicated chaos[3] and complex mixed-mode oscillations[2] has been reported recently.

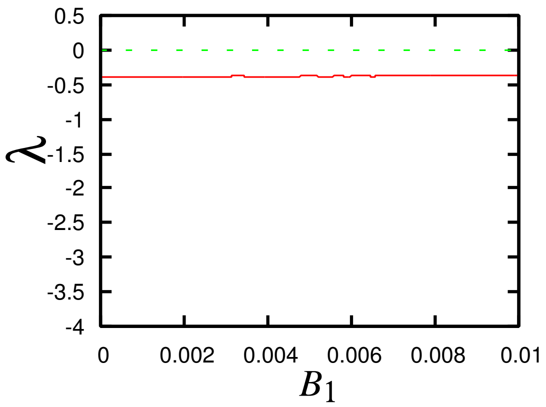
In this paper, we pay attention to chaos and oscillation death generated in Eq. (1) that Sekikawa *et al.* discovered[3]. The existence of oscillation death in this oscillator was first pointed out by them. They analyzed the disappearance of oscillation death and sudden emergence of chaotic oscillation that occurs around $B_1 = 0.04$. In this paper, we investigate the disappearance of chaotic oscillations and the appearance of oscillation death generated around $B_1 = 0.003$. Much smaller B_1 than that in discussed in [3] is investigated in this paper.

Since the forcing term is periodic, the Poincaré mapping is naturally defined as follows:

$$\begin{aligned} T_\sigma : \quad & R^2 \to R^2 \\ & (x, y)^\top \mapsto T_\sigma(x, y)^\top \equiv \varphi(2\pi/\omega;\ (x, y)^\top,\ \sigma), \end{aligned} \tag{2}$$



(a) One-parameter bifurcation diagram.



(b) Graph of the largest Lyapunov exponent.

Figure 3: One-parameter bifurcation diagram and the corresponding largest Lyapunov exponent where B_1 is varied($k_1 = 0.9$, $B_0 = 0.21$ and $\omega = 1.35$). The initial condition is $(x_0, y_0) = (0.555688, -0.384098)$ that is a stable equilibrium when no perturbation is applied.

where the superscript $\top$ denotes the transpose of the vector, $\varphi(\tau)$ is the solution, and σ is the bifurcation parameter.

Figure 2 (a) and (b) show a one-parameter bifurcation diagram of T_σ and the graph of the corresponding largest Lyapunov exponent λ. The largest lyapunov exponent is calculated by the algorithm proposed by Shimada and Nagashima[9]. The initial condition $(\tau, x, y) = (0, x_0, y_0)$ is chosen at the outer side of the stable relaxation oscillation that exists when no perturbation is applied, i.e., $(x_0,\ y_0) = (3, 3)$. Within the range of $0 < B_1 < B_c \simeq 0.001$, $\lambda = 0$. It means that the two-torus is stable for weak perturbation and that quasi-periodic attractor is generated in this range. Such a structure is widely observed in the driven oscillators[10]. A phase-locking occurs at $B_1 \simeq 0.00102$. From the graph of the largest Lyapunov exponent, it is understood that period-doubling cascades occur. Then, the largest Lyapunov exponent becomes positive, and chaos appears after the accumulation of the period-doubling cascades. Chaos can be observed near $B_1 = 0.002$. The generation of chaos for such small force is not rare in slow-fast dynamics[3]. The tendency becomes more conspicuous for smaller ε from our experience.

After the usual route to chaos, oscillation death appears. Oscillation death in this circuit has been first reported by Sekikawa *et al.*[3]. It is noteworthy that the stable relaxation oscillation that exists when no perturbation is applied, has already disappeared at $B_1 = 0.004$, and only oscillation death remains. The reason why the oscillations disappear for such extremely weak periodic perturbation, has not yet been clarified.

According to the numerical results, oscillation death always exists in the parameter range $B_1 \in [0, 0.01]$. The one-parameter bifurcation diagram and the corresponding graph of the largest Lyapunov exponent are presented in Fig. 3, where the initial condition is given by $(x_0, y_0) = (0.555688, -0.384098)$ which is a stable focus when no perturbation is applied. In Fig. 3, only the initial condition is different from the one of Fig. 2. Neither chaos nor relaxation oscillation is observed in Fig. 3. We can obtain no information and no traces about the disappearance of chaos mentioned above, from the one-parameter bifurcation diagram and the graph of the largest Lyapunov exponent shown in Fig. 3 as if no influence is generated to the state of oscillation death.

4. Basin of Oscillation Death

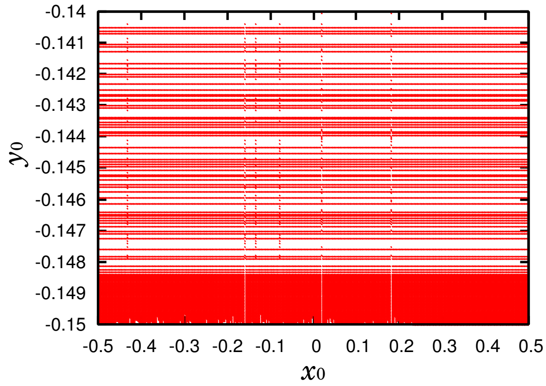
In this section, we derive a basin for oscillation death. Namely, we investigate the set of initial conditions whose attractor becomes oscillation death. We concentrate our attention to two cases: $B_1 = 0.002$ and $B_1 = 0.004$.

First, we present the case of $B_1 = 0.002$. As mentioned in the previous section, at least, chaotic attractor and oscillation death coexist. We investigate the initial state whose attractor is oscillation death. The points are plotted if $T_\sigma^{1,000}(x_0, y_0)$ converges to oscillation death, where $T_\sigma^{1,000}$ is a 1,000 times composite of T_σ. Figure 4 shows the basin. The condition near the boundaries are very complex. The patterns of the complex upper basin and the lower basin could be different.

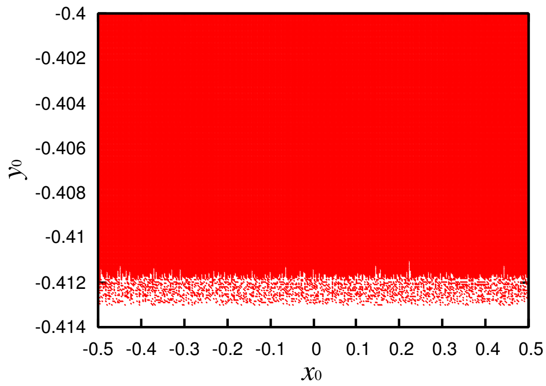
Figure 5 shows the basin for oscillation death with $B_1 = 0.004$. As the parameter B_1 is increased up to 0.004, all solutions which have any initial condition converge to oscillation death as far as numerical results are concerned. The entire surface are completely shaded, which is predicted by Figs. 2 and 3. However, we could not clarify what kind of bifurcations occur and how chaotic attractor disappears between $B_1 = 0.002$ and 0.004.

5. Concluding Remarks

We investigated a weakly driven Bonhoeffer-van der Pol oscillator where the parameter values were chosen such



(a)



(b)

Figure 4: Complicated basin (k_1 = 0.9, B_0 = 0.21 and ω = 1.35). (a) Upper part of the basin $x_0[-0.5, 0.5]$, $y_0[-0.15, -0.14]$ Grid mesh:500 × 500. (b) Lower part of the basin $x_0[-0.5, 0.5]$, $y_0[-0.414, -0.4]$ Grid mesh:500 × 500.

that a stable focus and a stable relaxation oscillation coexisted in close proximity when no perturbation was applied. Chaotic behavior was observed when the forcing term was small. However, this chaos was submerged by oscillation death although the periodic perturbation was very weak. The basin of oscillation death was investigated. When oscillation death and chaos coexist, the boundary of the basin is remarkably complex. However, when B_1 = 0.004, any solution converged to oscillation death. We could not understand how chaotic oscillations disappeared and how the solution arrived at the oscillation death. To clarify the mechanism of the disappearance of chaos under extremely weak perturbation is our interesting future research topic.

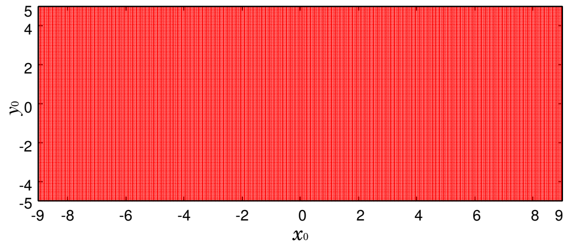


Figure 5: Basin of oscillation death. (k_1 = 0.9, B_0 = 0.21, ω = 1.35 and B_1 = 0.004).

References

[1] A. Rabinovitch and I. Rogachevskii, "Threshold, excitability and isochrones in the Bonhoeffer-van der Pol system," Chaos, Vol. 9, pp. 880–886, 1999.

[2] K. Shimizu, M. Sekikawa, and N. Inaba, "Mixed-mode oscillations and chaos from a simple second-order oscillator under weak periodic perturbation," Phys. Lett. A, Vol. 375, pp. 1566–1569, 2011.

[3] M. Sekikawa, K. Shimizu, N. Inaba, H. Kita, T. Endo, K. Fujimoto, T. Yoshinaga, and K. Aihara, "Sudden change from chaos to oscillation death in the Bonhoeffer-van der Pol oscillator," Phys. Rev. E, Vol. 84, pp. 056209-1–8, 2011.

[4] B. Braaksma and J. Grasman, "Critical dynamics of the Bonhoeffer-van der Pol equation and its chaotic response to periodic stimulation," Physica D, Vol. 68, pp. 265–280, 1993.

[5] T. Nomura, S. Sato, S. Doi, JP. Segundo, and M. D. Stiber, "A Bonhoeffer-van der Pol oscillator model of locked and non-locked behaviors of living pacemaker neurons," Biol. Cybern, Vol. 69, pp. 429–437, 1993.

[6] A. Rabinovitch, R. Thieberger, and M. Friedman, "Forced Bonhoeffer-van der Pol oscillator in its excited mode," Phys. Rev. E, Vol. 50, pp. 1572–1578, 1994.

[7] T. Nomura, S. Sato, S. Doi, J. P. Segundo, and M. D. Stiber, "Global bifurcation structure of a Bonhoeffer-van der Pol oscillator driven by periodic pulse trains," Biol. Cybern, Vol. 72, pp. 55–67, 1994.

[8] S. Doi and S. Sato, "Mathematical Biosciences, The global bifurcation structure of the BVP neuronal model driven by periodic pulse trains," Math. Biosci, Vol. 125, pp. 229–250, 1995.

[9] I. Shimada and T. Nagashima, "A numerical approach to Ergordic problem of dissipative dynamical systems," Prog. Theor. Phys., Vol. 61, pp. 1605–1615, 1979.

[10] N. Inaba and S. Mori, "Chaos via Torus Breakdown in a Piecewise-Linear Forced van der Pol Oscillator with a Diode" IEEE Trans. Circuit Syst., Vol. CAS-38, pp. 398–409, 1991.