IEICE Proceeding Series

A Method for the Computation of Border Collision Bifurcation Point in a Piecewise Linear System with Interrupted Characteristics

Yusuke Tone, Hiroyuki Asahara, Tetsushi Ueta, Kazuyuki Aihara, Takuji Kousaka

Vol. 1 pp. 828-831
Publication Date: 2014/03/17
Online ISSN: 2188-5079

2012 International Symposium on Nonlinear Theory and its Applications
NOLTA2012, Palma, Majorca, Spain, October 22-26, 2012

A Method for the Computation of Border Collision Bifurcation Point in a Piecewise Linear System with Interrupted Characteristics

Yusuke Tone†, Hiroyuki Asahara†, Tetsushi Ueta*, Kazuyuki Aihara‡ and Takuji Kousaka†

†Department of Mechanical and Energy Systems Engineering,
Oita University, 700 Dannoharu, Oita, 870-1192 Japan
∗ Center for Admin. Info. Tech. The University of Tokushima,
2-1 Minamijyosanjima-cho, Tokushima-shi, 770-8506 Japan
‡Institute of Industrial Science,
The University of Tokyo, 4-61 Komaba, Meguro-ku, Tokyo, 153-8505 Japan
Email: yusuke@bifurcation.jp, asahara@bifurcation.jp,
tetsushi@is.tokushima-u.ac.jp, aihara@sat.t.u-tokyo.ac.jp, takuji@oita-u.ac.jp

Abstract—This paper proposes a simple method for analyzing the border-collision bifurcation point in a piecewise linear interrupted system. First, we show the system dynamics, and then we derive the Poincaré map. Next, we explain the method for calculating the border-collision bifurcation point based on the Poincaré map approach. Finally, we apply the method to the current mode controlled buck-boost converter. The analysis result will show the validity of the method.

1. Introduction

The piecewise smooth systems (PWS) have a interrupted characteristic due to the complex switching actions. There are many papers that have studied the bifurcation phenomena in the PWS. We know that it is important to derive the bifurcation sets for understanding the qualitative property of the PWS. The calculation method for the local bifurcations, such as the period-doubling bifurcation and Neimark-Sacker bifurcation, have been studied. On the other hand, a new non-local bifurcation phenomenon, called the border-collision bifurcation, has been discovered in the PWS [1]. Many papers have studied mechanism of the border-collision bifurcation [2, 3]. In addition, some papers have proposed the calculation method of the border-collision bifurcation point based on the Poincaré map approach [4].

The sampled data model of the interrupted electric circuit, whose switch is dependent on the state and a periodic interval, often constructs the PWS. It means that the border-collision bifurcation can occur in the interrupted electric circuit, and indeed Refs. [5, 6] have reported appearance of the border-collision bifurcation. Of course there is an algorithm for calculating the bifurcation point of the border-collision [7]. In theory, we can calculate the bifurcation point by using above algorithm even if the system has nonlinear characteristics and is high-dimensional. But, the algorithm [7] is complicated because it needs to analytically derive the composite Poincaré map and its derivative. For this reason, it is valuable to construct the algorithm with no above complicated computation process.

In this paper, we aim to construct a simple algorithm for calculating the bifurcation point of the border-collision with the help of the mathematical tool *MAPLE*; the *MAPLE* will help to derive the composite Poincaré map and its derivative even if the system is high-dimensional [8]. First, we show the algorithm based on the Poincaré map approach. Next, we apply the method to the current mode controlled buck-boost converter. It will be possible to calculate the bifurcation point of the high-dimensional system by using the proposed algorithm. But, we apply the algorithm for the two-dimensional buck-boost converter as the first step to confirm the validity of the algorithm. Finally, we show that the algorithm can calculate the bifurcation point of the border-collision.

2. Bifurcation analysis

2.1. System description

We consider the following differential equations:

$$\begin{cases} \dfrac{dx}{dt} = f_{\mathrm{a}}(x, y, \boldsymbol{\lambda}_{\mathrm{a}}, \boldsymbol{\lambda}_{\mathrm{c}}) \\ \dfrac{dy}{dt} = g_{\mathrm{a}}(x, y, \boldsymbol{\lambda}_{\mathrm{a}}, \boldsymbol{\lambda}_{\mathrm{c}}) \end{cases}, \text{ System-a,} \tag{1}$$

$$\begin{cases} \dfrac{dx}{dt} = f_{\mathrm{b}}(x, y, \boldsymbol{\lambda}_{\mathrm{b}}, \boldsymbol{\lambda}_{\mathrm{c}}) \\ \dfrac{dy}{dt} = g_{\mathrm{b}}(x, y, \boldsymbol{\lambda}_{\mathrm{b}}, \boldsymbol{\lambda}_{\mathrm{c}}) \end{cases}, \text{ System-b,} \tag{2}$$

where $\boldsymbol{\lambda}_{\mathrm{a}}, \boldsymbol{\lambda}_{\mathrm{b}} \in \boldsymbol{R}^r$ and $\boldsymbol{\lambda}_{\mathrm{c}} \in \boldsymbol{R}^s$ are the system parameters. Let the exact solutions of Eqs. (1) and (2) be:

$$\begin{cases} x(t) = \varphi_{\mathrm{a}}(t; x, y, \boldsymbol{\lambda}_{\mathrm{a}}, \boldsymbol{\lambda}_{\mathrm{c}}) \\ y(t) = \phi_{\mathrm{a}}(t; x, y, \boldsymbol{\lambda}_{\mathrm{a}}, \boldsymbol{\lambda}_{\mathrm{c}}) \end{cases}, \text{ System-a,} \tag{3}$$

$$\begin{cases} x(t) = \varphi_{\mathrm{b}}(t; x, y, \boldsymbol{\lambda}_{\mathrm{b}}, \boldsymbol{\lambda}_{\mathrm{c}}) \\ y(t) = \phi_{\mathrm{b}}(t; x, y, \boldsymbol{\lambda}_{\mathrm{b}}, \boldsymbol{\lambda}_{\mathrm{c}}) \end{cases}, \text{ System-b.} \tag{4}$$

A local section Π in the state space is described as follows:

$$\Pi = \left\{ \boldsymbol{x} \in \boldsymbol{R}^2 \mid q(y) = 0,\ q : \boldsymbol{R}^2 \to \boldsymbol{R} \right\}. \tag{5}$$

Note that the local section Π is a scalar function whose is dependent on the state variable y.

Figure 1 shows a conceptual diagram for understanding behavior of the waveforms. Let the initial waveform at $t = kT$ be $\boldsymbol{x}_k = \boldsymbol{x}(kT) = (x_k, y_k)^\top$ with System-a. We assume that there are two types of the waveform behaviors during a time interval T. The one keeps System-a during a time interval T. The other one keeps System-a during a time $\tau_a(\boldsymbol{x}_k)$, and then the System keeps System-b during a time $T - \tau_a(\boldsymbol{x}_k)$.

At the next stage, we derive the Poincaré map for the following analysis. A map M_{Ak} is defined by the following equation if $\tau_a \geq T$ is satisfied.

$$M_{Ak} : \boldsymbol{R}^2 \to \boldsymbol{R}^2$$
$$\left\{ \begin{array}{l} x_k \longmapsto x_{k+1} = \varphi_a(T, x_k, y_k, \lambda_a, \lambda_c) \\ y_k \longmapsto y_{k+1} = \phi_a(T, x_k, y_k, \lambda_a, \lambda_c) \end{array} \right. . \tag{6}$$

Likewise, a map M_{Bk} is defined by the following equation if $\tau_a < T$ is satisfied.

$$\begin{array}{l} M_{Bk} : \boldsymbol{R}^2 \to \boldsymbol{R}^2 \\ \quad \boldsymbol{x}_k \longmapsto \boldsymbol{x}_{k+1} = M_b \circ M_a. \end{array} \tag{7}$$

The maps M_a and M_b is defined as follows:

$$M_a : \boldsymbol{R}^2 \to \prod$$
$$\left\{ \begin{array}{l} x_k \longmapsto x_q = \varphi_a(\tau_a, x_k, y_k, \lambda_a, \lambda_c) \\ y_k \longmapsto y_q = \phi_a(\tau_a, x_k, y_k, \lambda_a, \lambda_c) \end{array} \right. , \tag{8}$$

$$M_b : \prod \to \boldsymbol{R}^2$$
$$\left\{ \begin{array}{l} x_q \longmapsto x_{k+1} = \varphi_b\left(T - \tau_a, x_q, y_q, \lambda_b, \lambda_c\right) \\ y_q \longmapsto y_{k+1} = \phi_b\left(T - \tau_a, x_q, y_q, \lambda_b, \lambda_c\right) \end{array} \right. . \tag{9}$$

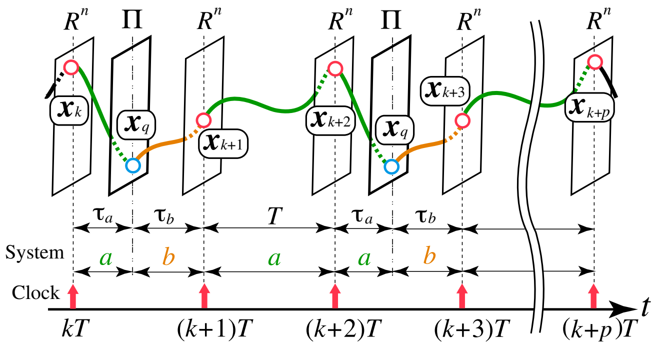


Figure 1: Example of behavior.

Based on Eqs. (6) and (7), let the exact Poincaré map be

$$\begin{aligned} \boldsymbol{x}_{k+1} &= \begin{bmatrix} F(1, x_k, y_k, \lambda_a, \lambda_b, \lambda_c) \\ G(1, x_k, y_k, \lambda_a, \lambda_b, \lambda_c) \end{bmatrix} \\ &= \left\{ \begin{array}{ll} \begin{bmatrix} F_A(x_k, y_k, \lambda_a, \lambda_c) \\ G_A(x_k, y_k, \lambda_a, \lambda_c) \end{bmatrix}, & y_k \leq D \\ \begin{bmatrix} F_B(x_k, y_k, \lambda_a, \lambda_b, \lambda_c) \\ G_B(x_k, y_k, \lambda_a, \lambda_b, \lambda_c) \end{bmatrix}, & y_k > D \end{array} \right. , \end{aligned} \tag{10}$$

where a border D is

$$D = \phi_a(-T, x_{k+1}, y_{k+1}, \lambda_a, \lambda_c). \tag{11}$$

2.2. Algorithm for calculating bifurcation point

The conditional equation for the period-p solution is defined as follows:

$$\begin{bmatrix} F(p, x_k, y_k, \lambda_a, \lambda_b, \lambda_c) - x_k \\ G(p, x_k, y_k, \lambda_a, \lambda_b, \lambda_c) - y_k \end{bmatrix} = \mathbf{0}. \tag{12}$$

We assume that the border-collision bifurcation occurs when a solution y_k just on the border D. This situation is expressed as follows:

$$y_k - D = 0. \tag{13}$$

Based on Eqs. (12) and (13), we get

$$P(x_k, y_k, \lambda_{c1}) = \begin{bmatrix} F(p, x_k, y_k, \lambda_{c1}) - x_k \\ G(p, x_k, y_k, \lambda_{c1}) - y_k \\ y_k - D \end{bmatrix} = \mathbf{0}, \tag{14}$$

where a parameter λ_{c1} belongs to a parameter λ_c. Although the previous algorithm numerically calculates Eq. (14) **??**, the proposed algorithm can derive Eq. (14) with the exact solution. But, we need the numerical computation then for calculating the bifurcation point of border collision. The Jacobian matrix is derived based on Eq. (14). Specifically, we get

$$DP(x_k, y_k, \lambda_{c1}) = \begin{bmatrix} \dfrac{\partial F}{\partial x_k} - 1 & \dfrac{\partial F}{\partial y_k} & \dfrac{\partial F}{\partial \lambda_{c1}} \\ \dfrac{\partial G}{\partial x_k} & \dfrac{\partial G}{\partial y_k} - 1 & \dfrac{\partial G}{\partial \lambda_{c1}} \\ 0 & 1 & \dfrac{\partial \phi_a}{\partial \lambda_{c1}} \end{bmatrix}. \tag{15}$$

3. Example of the application

We apply the method to the current mode controlled buck-boost converter (see Fig. 2). Note that we only consider the continuous conduction mode for the sake of the simplicity. Moreover, we use the dimensionless values: $y = \sqrt{L}i$, $x = \sqrt{C}v$, $\tau = t/\sqrt{LC}$, $\sigma = \sqrt{L/C}/(2R)$, and $B = \sqrt{C}E$. The circuit equations are derived as follows:

$$\left\{ \begin{array}{l} \dfrac{dx}{d\tau} = -2\sigma x \\ \dfrac{dy}{d\tau} = B \end{array} \right. , \ \text{SW : ON}, \tag{16}$$

$$\begin{cases} \dfrac{dx}{d\tau} = y - 2\sigma x \\ \dfrac{dy}{d\tau} = -x \end{cases}, \text{ SW : OFF.} \tag{17}$$

Figure 3 shows the behavior of the waveforms. The Poincaré map is derived with the exact solution.

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{cases} \begin{bmatrix} x_k e^{2\alpha T} \\ y_k + BT \end{bmatrix}, & y_k \le D \\ \begin{bmatrix} e^{\alpha\tau_b}\Big[x_k e^{2\alpha\tau_a}\cos\beta\tau_b \\ \quad + \dfrac{1}{\beta}\Big\{\alpha x_k e^{2\alpha\tau_a} \\ \qquad + \left(\alpha^2+\beta^2\right) y_{\rm ref}\Big\}\sin\beta\tau_b\Big] \\ e^{\alpha\tau_b}\Big[y_{\rm ref}\cos\beta\tau_b + \dfrac{1}{\beta}\Big\{x_k e^{2\alpha\tau_a} \\ \quad +\alpha y_{\rm ref}\Big\}\sin\beta\tau_b\Big] \end{bmatrix}, & y_k > D \end{cases}, \tag{18}$$

where $\alpha = -\sigma$, $\beta = \sqrt{\sigma^2 - 1}$, and $\tau_a(y_k) = (y_{\rm ref} - y_k)/B$, respectively. τ_b is expressed as follows:

$$\tau_b(y_k) = T\left[1 - \left(\frac{\tau_a}{T} \bmod 1\right)\right]. \tag{19}$$

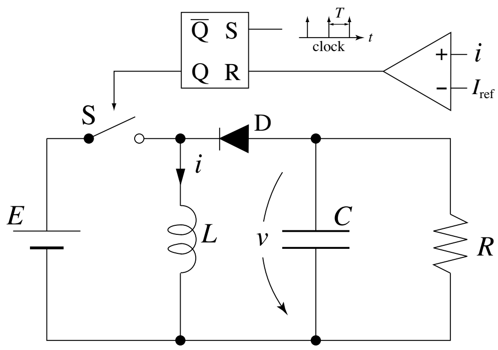


Figure 2: Buck-boost converter.

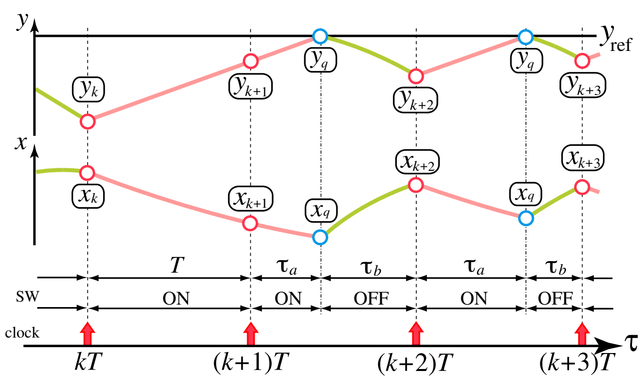


Figure 3: Behavior of the buck-boost converter.

In addition, the border D is given by

$$D = y_{\rm ref} - BT. \tag{20}$$

Note that we omit the detailed expressions for the behavior of the waveforms and for the derivation of the Poincaré map because they have reported in many previous papers [9,10].

In the following analysis, we fix the parameters σ and T as $\sigma = 0.2795$ and $T = 1.1180$ [11]. Figure 4 shows an example of the 1 parameter bifurcation diagram upon varying the bifurcation parameter B from $B = 0.0$ to $B = 0.1$. We observe the period-doubling bifurcation and the border-collision bifurcation in the figure. For example, the period-1 solution bifurcates to the period-2 solution around $B = 0.08$ through the period-doubling bifurcation. On the other hand, it is observed that the border-collision bifurcation occurs and the period-2 solution bifurcates to the period-4 solution around $B = 0.05$; because a part of the period-2 solution collides with the border D and period-4 solution is generated. Specifically, Fig. 5 shows the examples of the waveforms and the corresponding phase plane. This figure corresponds to the parameters (a) and (b) in Fig. 4. It is also clear from Fig. 5 that a part of the period-2 solution bifurcates to the border D and the period-4 solution is generated. So, we focused on this border-collision bifurcation, and applied the proposed algorithm. Table 1 shows the calculation results. The table says that the border-collision bifurcation appears around $B = 0.052$ when $y_{\rm ref} = 0.08$. The bifurcation point $B = 0.052$ is correct in the 1 parameter bifurcation diagram (see 4). Therefore, we conclude that the proposed algorithm can calculate the bifurcation point of the border-collision.

4. Conclusion

In this paper, we have presented an algorithm for calculating the bifurcation point of the border-collision in the piecewise linear system based on the Poincaré map approach. First, we explained the system dynamics. Then, we constructed the algorithm with the exact solution. Finally, we applied the algorithm for the current mode controlled buck-boost converter and calculated the bifurcation point of the border-collision. The proposed algorithm do not need to calculate the Poincaré map and its derivative numerically. We know that the numerical computation processes for deriving the Poincaré map and its derivative is complicated. Therefore, we believe that the proposed algorithm is valuable to calculate the bifurcation point of the border-collision in the high-dimensional system. Moreover, the mathematical tool *MAPLE* will help us to derive the exact Poincaré map and its derivative in the high-dimensional systems. In future, we will apply the algorithm to the high-dimensional systems.

Acknowledgement

This research is partially supported by the Aihara Innovative Mathematical Modelling Project, the Japan Society for the Promotion of Science (JSPS) through the "Funding Program for World-Leading Innovative R&D on Science and Technology (FIRST Program)," initiated by the Council for Science and Technology Policy (CSTP).

Table 1: Calculation results ($\sigma = 0.2795$, $T = 1.1180$).

x	y	B	y_{ref}	Remarks
0.0660	0.0246	0.0585	0.090	BC
0.0623	0.0232	0.0552	0.085	BC
0.0587	0.0219	0.0520	0.080	BC
0.0550	0.0205	0.0487	0.075	BC
0.0513	0.0191	0.0455	0.070	BC

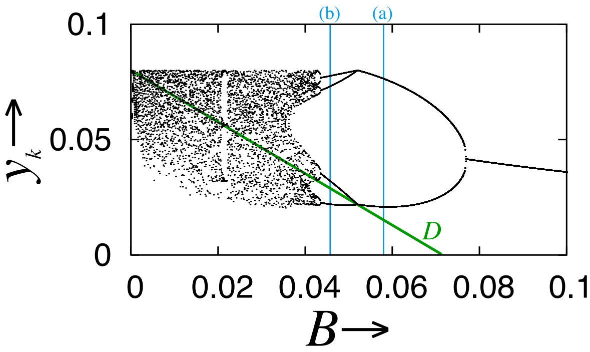
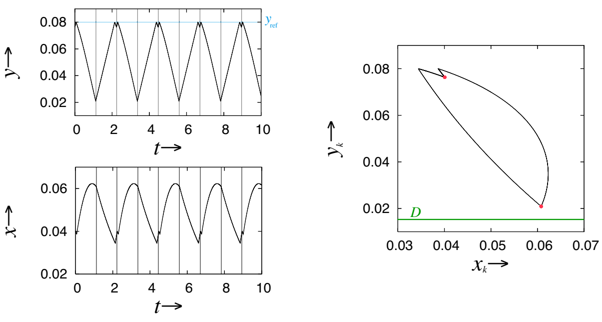
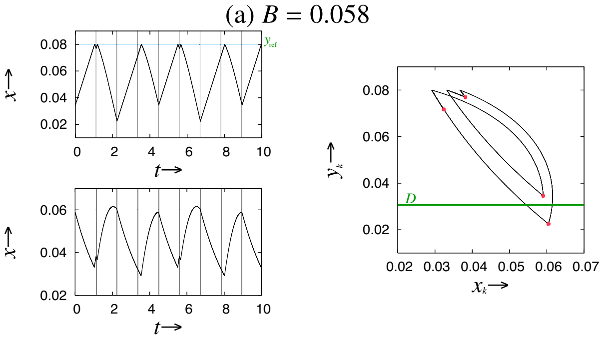


Figure 4: 1 parameter bifurcation diagram ($y_{\mathrm{ref}} = 0.08$).



(a) $B = 0.058$



(b) $B = 0.044$

Figure 5: Waveforms and phase portraits ($y_{\mathrm{ref}} = 0.08$).

References

[1] H. E. Nusse, J. A. York, "Border-Collision Bifurcations Including "Period Two to Period Three" for Piecewise Smooth System", PHYSICA D, Vol. 57, pp. 39–57, 1992.

[2] L. Gardini, D. Fournier-Prunaret, P. Charg'e, "Border collision bifurcations in a two-dimensional piecewise smooth map from a simple switching", Chaos, an interdisciplinary journal of nonlinear science, Vol. 21, No. 2, 2011.

[3] D. Fournier-Prunaret, P. Charg'e, L. Gardini, "Border collision bifurcations and chaotic sets in a two-dimensional piecewise linear map", Communications in Nonlinear Science and Numerical Simulation, Vol. 16, No. 2, pp. 916–927, 2011.

[4] K. Kinoshita, T. Ueta, J. Imura, K. Aihara, "Bifurcation and Synchronization Phenomena of Coupled Nagumo-Sato Models", Proc. NCSP, pp. 599–602, 2012.

[5] S. Banerjee, P. Ranjan, C Grebogi, "Bifurcations in Two-Dimensional Piecewise Smooth Maps–Theory and Applications in Switching Circuits", Proc. IEEE, Vol. 47, No. 5, pp. 633–643, 2000.

[6] P. Jain, S. Banerjee, "Border-Collision Bifurcations in One-Dimensional Discontinuous Maps", International Journal of Bifurcation and Chaos, Vol. 13, No. 11, pp. 3341–3351, 2003.

[7] T. Kousaka, T. Ueta, Y. Ma, H. Kawakami, "Bifurcation analysis of a piecewise smooth system with nonlinear characteristics", International Journal of Circuit Theory and Applications, Vol. 33, pp. 263–279, 2005.

[8] www.maplesoft.com

[9] S. Banerjee, GC. Berghese, "Nonlinear phenomena in power electronics: attractors, bifurcations", Chaos, and Nonlinear Control IEEE Press 2001.

[10] CK. Tse, "Complex Behavior of Switching Power Converters", CRC Press 2003.

[11] K. W. E. Cheng et al, "Chaos study and parameter-space analysis of the DC–DC buck-boost converter", IEE Proc-Electric Power Appl, Vol. 150, pp. 126–138, 2003.