IEICE Proceeding Series

A computer-assisted proof method of the invertibility to elliptic operators

Akitoshi Takayasu, Shin'ichi Oishi

Vol. 1 pp. 816-819
Publication Date: 2014/03/17
Online ISSN: 2188-5079

2012 International Symposium on Nonlinear Theory and its Applications
NOLTA2012, Palma, Majorca, Spain, October 22-26, 2012

A computer-assisted proof method of the invertibility to elliptic operators

Akitoshi Takayasu† and Shin'ichi Oishi‡

†Faculty of Science and Engineering, Waseda University
‡Faculty of Science and Engineering, Waseda University & CREST/JST
3-4-1, Okubo, Shinjuku, Tokyo, 169-8555 Japan
Email: takitoshi@aoni.waseda.jp, oishi@waseda.jp

Abstract—In this talk, a computer-assisted analysis procedure is proposed with respect to the invertibility of some elliptic operators. Based on a verified eigenvalue evaluation for the Laplace operator, the inverse of an elliptic operator is proved with computer-assistance. Whether the operator has its inverse plays important role in computer-assisted proof methods for nonlinear elliptic problems. The invertibility of considered operator is related to some shifted eigenvalue or weighted eigenvalue problems. A computer-assisted analysis method is proposed in this talk. Furthermore, some applications are presented for semilinear elliptic problems.

1. Framework of verified computations for semilinear partial differential equations

Let Ω be bounded polygonal domain in $\mathbb{R}^2$ with arbitrary shape. In this talk, let us be concerned with Dirichlet boundary value problem of the semilinear elliptic equation:

$$\begin{cases} -\Delta u = f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $f : H_0^1(\Omega) \to L^2(\Omega)$ is assumed to be Fréchet differentiable. For example,

$$f(u) = c_1 u + c_2 u^2 + c_3 u^3 + ... + c_p u^p + g$$

with $c_i \in L^\infty(\Omega)$, $(i = 1, ..., p < \infty)$ and $g \in L^2(\Omega)$ satisfies this condition. Verified computation approach will be adopted to explore the existence and local uniqueness of weak solution of (1). Namely, if an approximate solution is given by certain numerical method, we will try to validate the existence of exact solution in the neighbourhood of the approximation. In the classical analysis of variational theory, weak solution of Dirichlet boundary problem (1) is defined in the variational form:

$$\begin{aligned} &\text{Find } u \in H_0^1(\Omega), \quad \text{satisfying} \\ &(\nabla u, \nabla v) = (f(u), v), \quad \text{for all } v \in H_0^1(\Omega). \end{aligned}$$

Here, let us define

$$(\nabla u, \nabla v) := \int_\Omega \nabla u \cdot \nabla v dx$$

and

$$(f(u), v) := \int_\Omega f(u) v dx.$$

Now we put $V = H_0^1(\Omega)$ for simple formulation. Let us define linear and nonlinear operators $\mathcal{A}, \mathcal{N} : V \to V$,

$$(\mathcal{A}u, v)_V := (\nabla u, \nabla v),$$

$$(\mathcal{N}(u), v)_V := (f(u), v).$$

Furthermore, we define $\mathcal{F} : V \to V$ as

$$\mathcal{F}(u) := \mathcal{A}u - \mathcal{N}(u).$$

The original problem (1) is transformed into the following nonlinear operator equation:

$$\text{Find } u \in V, \quad \text{satisfying } \mathcal{F}(u) = 0 \text{ in } V. \tag{2}$$

$\mathcal{F} : V \to V$ is assumed to be the Fréchet differentiable mapping. Let $\hat{u} \in V_h \subset V$ be an approximate solution to (2). Fréchet derivative of $\mathcal{F}$ at $\hat{u}$ is denoted by $\mathcal{F}'[\hat{u}] : V \to V$. *i.e.* satisfying

$$\|\mathcal{F}(\hat{u} + v) - \mathcal{F}(\hat{u}) - \mathcal{F}'[\hat{u}]v\|_V = o(\|v\|_V), \ \|v\|_V \to 0.$$

In order to verify the existence and local uniqueness of the exact solution in the neighborhood of $\hat{u}$, we consider to apply the Newton-Kantorovich theorem [1] to (2).

Theorem 1 (Newton-Kantorovich)
Assuming the Fréchet derivative $\mathcal{F}'[\hat{u}] : V \to V$ is nonsingular and satisfies

$$\|\mathcal{F}'[\hat{u}]^{-1}\mathcal{F}(\hat{u})\|_V \le \alpha,$$

for a certain positive α. Then, let

$$\overline{B}(\hat{u}, 2\alpha) := \{v \in V : \|v - \hat{u}\|_V \le 2\alpha\}$$

be a closed ball centered at $\hat{u}$ with radius 2α. Let also $D \supset \overline{B}(\hat{u}, 2\alpha)$ be an open ball in V. We assume that for a certain positive ω, it holds:

$$\|\mathcal{F}'[\hat{u}]^{-1}(\mathcal{F}'[v] - \mathcal{F}'[w])\|_{V,V} \le \omega\|v - w\|_V, \quad \forall v, w \in D.$$

If $\alpha\omega \le \frac{1}{2}$ holds, then there is a solution $u \in V$ of (2) satisfying

$$\|u - \hat{u}\|_V \le \rho := \frac{1 - \sqrt{1 - 2\alpha\omega}}{\omega}.$$

Furthermore, the solution u is unique in $\overline{B}(\hat{u}, \rho)$.

Remark 1 *To apply the Newon-Kantorovich theorem, we will calculate three constants below explicitly.*

$$\|\mathcal{F}'[\hat{u}]^{-1}\|_{V,V} \le C_1, \tag{3}$$

$$\|\mathcal{F}(\hat{u})\|_V \le C_{2,h},$$

$$\|\mathcal{F}'[v] - \mathcal{F}'[w]\|_{V,V} \le C_3\|v-w\|_V, \quad \forall v, w \in D \subset V.$$

The constants above, C_1, $C_{2,h}$ and C_3, will yield

$$\|\mathcal{F}'[\hat{u}]^{-1}\mathcal{F}(\hat{u})\|_V \le C_1 C_{2,h},$$

and

$$\|\mathcal{F}'[\hat{u}]^{-1}(\mathcal{F}'[v] - \mathcal{F}'[w])\|_{V,V} \le C_1 C_3 \|v-w\|_V.$$

Therefore, if the condition $C_1^2 C_{2,h} C_3 \le 1/2$ is confirmed by verified computations, then the existence and local uniqueness of the solution are proved numerically based on the Newton-Kantorovich theorem.

2. Inverse of linearized operator

This part is devoted for the inverse norm estimation of linearized operator (3), which plays important role on the verified computation for semilinear equations. The invertibility of the Fréchet derivative $\mathcal{F}'[\hat{u}] : V \to V$ is led by the invertibility of an elliptic operator with respect to the following linear elliptic equations:

$$\begin{cases} -\Delta u + cu = g, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $c(x) \in L^\infty(\Omega)$ and $g(x) \in L^2(\Omega)$. The weak form of this problem is given by

$$\text{Find } u \in V, \text{ satisfying } (\nabla u, \nabla v) + (cu, v) = (g, v), \quad \forall v \in V.$$

The linear elliptic operator $\mathcal{L} : V \to V$ can be defined by

$$(\mathcal{L}u, v)_V := (\nabla u, \nabla v) + (cu, v), \quad \forall v \in V.$$

Let $f'[\hat{u}]$ be the Fréchet derivative of f at $\hat{u}$. Putting $c(x) := -f'[\hat{u}]$ as a $L^\infty(\Omega)$ function, we obtain $\mathcal{L} = \mathcal{F}'[\hat{u}]$. In the following, we would like to introduce a computer-assisted analysis procedure to show the solvability of the problem (4). There are some existing methods for this estimation. M. Plum uses the eigenvalue evaluation with his original homotopy technique [2]. Using the perturbation theorem of linear operator, S. Oishi evaluates the inverse of linearized operator for ODEs. Further M.T. Nakao, K. Hashimoto and Y. Watanabe [3] is evaluated the same value based on the fixed point theorem. This talk uses the eigenvalue evaluation the same as M. Plum's procedure. One feature of our method is that it doesn't use homotopy technique. We try to bound the weighted eigenvalue directly.

2.1. Weighted eigenvalue problem

Let σ be the real number satisfying

$$\sigma \ge \max_{x\in\Omega} c(x).$$

The 'σ-inner product' and its norm are defined by

$$(u, v)_\sigma := (\nabla u, \nabla v) + \sigma(u, v)$$

and

$$\|u\|_\sigma := \sqrt{(u, u)_\sigma}.$$

V is also Hilbert space with its inner product $(\cdot,\cdot)_\sigma$. For the linearized inverse norm estimation, the following property is obtained.

Lemma 1 *Assuming the set of real numbers satisfies*

$$\mathcal{M} = \{\, \mu \in \mathbb{R} \ : \ \exists u \in V \text{ satisfying } (\nabla u, \nabla v) + (c(x)u, v) = \mu(u, v)_\sigma, \ v \in V\}.$$

If $0 \notin \mathcal{M}$ holds, then there exists $\mathcal{L}^{-1}$ such that

$$\|\mathcal{L}^{-1}\|_{V,V} \le \max_{\mu\in\mathcal{M}} |\mu|^{-1}.$$

Proof. $\mathcal{M}$ is the spectrum of the elliptic operator $\mathcal{L}$. If $0 \notin \mathcal{M}$, then 0 is in the resolvent set, which means $\mathcal{L}$ has its inverse. Moreover,

$$\|\mathcal{L}^{-1}\|_{V,V} = \sup_{u\in V} \frac{\|u\|_V}{\|\mathcal{L}u\|_V} \le \max_{\mu\in\mathcal{M}} |\mu|^{-1}.$$

Proposal method evaluates the absolute minimum eigenvalue of $\mathcal{M}$, which is isolated with finite multiplicity. Let us transform the eigenvalue problem into the following form.

$$(\nabla u, \nabla v) + (c(x)u, v) = \mu(u, v)_\sigma$$
$$\Longleftrightarrow \ (u, v)_\sigma = \frac{1}{1-\mu}(a^2 u, v),$$

where $a(x) := \sqrt{\sigma - c(x)}$. Putting

$$\lambda = \frac{1}{1-\mu},$$

we have a weighted eigenvalue problem: Find $u \in V$ and $\lambda \in \mathbb{R}$ such that

$$(u, v)_\sigma = \lambda(a^2 u, v), \quad \forall v \in V. \tag{5}$$

2.2. Verified eigenvalue bounds

In order to bound the absolute value of μ neighboring 0, we will find the minimizer of $|1 - \lambda^{-1}|$ using eigenvalues of (5). If the λ includes 1 in its interval, μ is expected to contain 0. Then, the verification is failed based on the Lemma 1. On the basis of verified evaluation technique for Laplace operator [4], we obtain the verified bound of the weighted eigenvalue evaluation. First of all, the upper bound of desired eigenvalues are obtained by the Rayleigh-Ritz method. Let V_h be the finite dimensional subspace of V spanned by the finite element base functions $\{\phi_i\}_{i=1,\dots,N}$ for given $N \in \mathbb{N}$. S and A denote matrices whose i-j elements are given by

$$S_{ij} := (\phi_i, \phi_j)_\sigma, \ A_{ij} := (a^2\phi_i, \phi_j).$$

Then $\lambda_1^h < \lambda_2^h < \dots < \lambda_N^h$ denote eigenvalues of the generalize eigenvalue problem: $S\mathrm{x} = \lambda^h A\mathrm{x}$, which is bounded by verified numerical computations. The Rayleigh-Ritz bound gives

$$\lambda_k \le \lambda_k^h \quad \text{for } k = 1, \dots, N.$$

The lower bound is also needed. It is more difficult work than the upper bound. M. Plum [2] has proposed the procedure mixing the homotopy technique and the Lehmann-Goerisch method to obtain the lower bound. In this talk, we propose a method of obtaining the lower bound using the error analysis of an orthogonal projection. This method have a feature that we will bound eigenvalues of the weighted eigenvalue problem directly. This work is inspired by the previous work shown by X. Liu and S. Oishi [4]. Let $\mathcal{P}_h : V \to V_h$ be the orthogonal projection defined by

$$(u - \mathcal{P}_h u, v_h)_\sigma = 0, \ \forall v_h \in V_h.$$

For given $g = -\Delta u + \sigma u \in L^2(\Omega)$, C_M denotes a positive constant satisfying the error estimate

$$\|u - \mathcal{P}_h u\|_\sigma \le C_M \|g\|_{L^2}. \tag{6}$$

The following main theorem of this work is introduced.

Theorem 2 *Let λ_k be the k-th eigenvalue of (5) whose approximation is denoted by λ_k^h with verification. If*

$$1 - \lambda_k \|a\|_\infty^2 C_M^2 > 0$$

holds, then the verified lower bound of eigenvalue

$$\frac{\lambda_k^h}{1 + \lambda_k^h \|a\|_\infty^2 C_M^2} \le \lambda_k$$

is obtained.

Proof. Let u_k be an eigenfunction corresponding to each λ_k with $\|au_k\|_{L^2} = 1$. Let E_k be the space spanned by eigenfunctioins $\{u_i\}_{i=1,\dots,k}$. For $\forall v = \sum_{i=1}^k d_i u_i \in E_k$, we have $\|av\|_{L^2} = 1$. (6) and Aubin-Nitsche's trick give

$$\|v - \mathcal{P}_h v\|_{L^2} \le C_M \|v - \mathcal{P}_h v\|_\sigma, \quad \forall v \in V.$$

From the definition of u_k, it follows

$$\lambda_k := \frac{\|u_k\|_\sigma^2}{\|au_k\|_{L^2}^2} = \|u_k\|_\sigma^2,$$

which implies $\|v\|_\sigma^2 / \|av\|_{L^2}^2 \le \lambda_k$ for $v \in E_k$. Now, the min-max principle follows

$$\begin{aligned}
\lambda_k^h &\le \max_{v\in E_k} \frac{\|\mathcal{P}_h v\|_\sigma^2}{\|a\mathcal{P}_h v\|_{L^2}^2} \\
&= \max_{v\in E_k} \frac{\|v\|_\sigma^2 - \|v - \mathcal{P}_h v\|_\sigma^2}{\|av + a\mathcal{P}_h v - av\|_{L^2}^2} \\
&= \max_{v\in E_k} \frac{\|v\|_\sigma^2 - \|v - \mathcal{P}_h v\|_\sigma^2}{\|av\|_{L^2}^2 + 2(av, a\mathcal{P}_h v - av)_{L^2} + \|a\mathcal{P}_h v - av\|_{L^2}^2} \\
&= \max_{v\in E_k} \frac{\lambda_k - \|v - \mathcal{P}_h v\|_\sigma^2}{1 + 2(av, a\mathcal{P}_h v - av)_{L^2} + \|a\mathcal{P}_h v - av\|_{L^2}^2} \\
&\le \max_{v\in E_k} \frac{\lambda_k - \|v - \mathcal{P}_h v\|_\sigma^2}{1 - 2\|a\mathcal{P}_h v - av\|_{L^2} + \|a\mathcal{P}_h v - av\|_{L^2}^2} \\
&= \max_{v\in E_k} \frac{\lambda_k - \|v - \mathcal{P}_h v\|_\sigma^2}{(1 - \|a\mathcal{P}_h v - av\|_{L^2})^2} \\
&\le \max_{v\in E_k} \frac{\lambda_k - \|v - \mathcal{P}_h v\|_\sigma^2}{(1 - \|a\|_\infty C_M \|v - \mathcal{P}_h v\|_\sigma)^2}.
\end{aligned}$$

Let

$$h(t) := \frac{\lambda_k - t^2}{(1 - \|a\|_\infty C_M t)^2}.$$

It is obtain that $h(t)$ is monotonically increasing if t satisfies $t \le \lambda_k \|a\|_\infty C_M$ and $1 - \|a\|_\infty C_M t > 0$. Here,

$$\begin{aligned}
\|v - \mathcal{P}_h v\|_\sigma &\le C_M \| - \Delta v + \sigma v\|_{L^2} \\
&= C_M \|\lambda_k a^2 v\|_{L^2} \\
&\le \lambda_k C_M \|a\|_\infty \|av\|_{L^2} \\
&= \lambda_k \|a\|_\infty C_M
\end{aligned}$$

holds and the assumption $1 - \lambda_k \|a\|_\infty^2 C_M^2 > 0$ follows. Thus,

$$\begin{aligned}
\lambda_k^h &\le \max_{v\in E_k} h(\|v - \mathcal{P}_h v\|_\sigma) \\
&\le \max_{v\in E_k} h(\lambda_k \|a\|_\infty C_M) \\
&= \frac{\lambda_k}{1 - \lambda_k \|a\|_\infty^2 C_M^2}.
\end{aligned}$$

From this inequality, it is easy to have the desired result.

Remark 2 *Through the proposed method gives the direct evaluation for lower bounds, it is a rough bound for verification. If we want to have more precise evaluation, the Lehmann-Goerisch method can be applicable. However, the Lehmann-Goerisch method requires the sufficiently smooth base function (e.g. C^1-smooth functions). When we choose P_3 or more higher oder finite element base functions, the accurate evaluation can be available.*

3. Computational results

Now, we will present a numerical result. All computations are carried out on Mac OS X, 2.26GHz Quad-Core Intel Xeon by using MATLAB 2011a with a toolbox for verified computations, INTLAB [5]. We also use the mesh generator Gmsh [6] or our own uniform mesh generator. For an application of our computer-assisted proof method, we treat a semilinear Dirichlet boundary value problem on $\Omega = (0,1) \times (0,1)$:

$$\begin{cases} -\Delta u = u^2, & \text{in } \Omega, \\ u(x) = 0, & \text{on } \partial\Omega. \end{cases} \tag{7}$$

Obviously, the Fréchet derivative of right-hand side is given by $f'(\hat{u}) = 2\hat{u}$. An approximate solution $\hat{u}$ is calculated by FEM with piecewise quadratic (P_2) finite elements on uniform mesh and non-uniform mesh, see Fig.1 and 2.

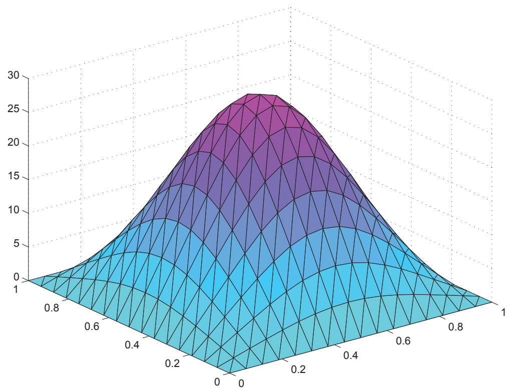

Fig.1: $\hat{u}$ of (7) on uniform mesh.

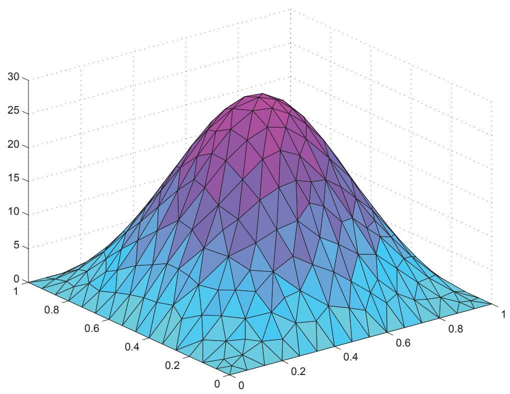

Fig.2: $\hat{u}$ of (7) on non-uniform mesh.

The verification of an elliptic operator is succeeded by proposal method and the existing method [3]. Comparing two methods, the proposed method is easy to verify the invertibility and accuracy is bit better. The weak condition for the verification is superiority of our proposed method.

In Table 1 and 2, the result of our proposal method for linearized inverse norm estimation is given. Here, λ^-, λ^+ are eigenvalues of (5) satisfying $\lambda^- < 1.0 < \lambda^+$. Their upper and lower bound of eigenvalues are bounded by the Rayleigh-Ritz bound and our main theorem (Theorem 2).

Table 1: Computational results on uniform mesh

$2^{-\eta}$	λ^-	λ^+	C_1	[3]
3	$0.^{501}_{484}$	$1.^{574}_{434}$	3.626	7.370
4	$0.^{501}_{496}$	1.5^{73}_{35}	2.938	3.363
5	$0.^{501}_{499}$	1.5^{73}_{62}	2.794	2.887
6	$0.^{501}_{499}$	1.57^{3}_{0}	2.759	2.782

Table 2: Computational results on non-uniform mesh

$2^{-\eta}$	λ^-	λ^+	C_1	[3]
3	0.4^{95}_{36}	$1.^{555}_{096}$	16.0651	Failed
4	0.4^{99}_{83}	$1.^{569}_{425}$	3.685	7.804
5	0.49^{9}_{4}	1.5^{72}_{24}	2.998	3.570
6	0.49^{9}_{8}	1.5^{73}_{60}	2.805	2.920

Acknowledgments

The authors would like to thank Dr. X. Liu in Waseda University for his fruitful suggestions and comments corresponding to verified eigenvalue evaluations. This work is deeply based on his previous work [4]. This research is supported by Research Fellow of the JSPS.

References

[1] L.V. Kantorovich and G.P. Akilov. *Functional analysis in normed spaces*. International series of monographs in pure and applied mathematics. Pergamon Press, 1964.

[2] M. Plum. Explicit H_2-estimates and pointwise bounds for solutions of second-order elliptic boundary value problems, Journal of Mathematical Analysis and Applications, 165, (1992) pp.36–61.

[3] M.T. Nakao, K. Hashimoto, and Y. Watanabe. A numerical method to verify the invertibility of linear elliptic operators with applications to nonlinear problems. *Computing*, 75(1), pp.1–14, 2005.

[4] X. Liu and S. Oishi. Verified eigenvalue evaluation for elliptic operator on arbitrary polygonal domain. in preparation.

[5] S.M. Rump. INTLAB - INTerval LABoratory. In Tibor Csendes, editor, *Developments in Reliable Computing*, pp. 77–104. Kluwer Academic Publishers, Dordrecht, 1999. `http://www.ti3.tu-harburg.de/rump/`.

[6] C. Geuzaine and J.-F. Remacle. Gmsh: A 3-d finite element mesh generator with built-in pre- and post-processing facilities. *International Journal for Numerical Methods in Engineering*, 79(11):1309–1331, 2009. `http://www.geuz.org/gmsh/`.