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On guaranteed eigenvalue estimation of compact differential operator with singularity

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Abstract—
In this paper, we consider the eigenvalue problem of elliptic operator $L := -\Delta u + \sigma u$ over 2D domain Ω:

$$\text{Find } u \in H_0^1(\Omega) \text{ and } \lambda \in \mathcal{R}\,, \quad Lu = \lambda u\,, \tag{1}$$

where $\sigma \in L_\infty(\Omega)$. In case of domain being polygonal one with re-entrant corner, the eigenfunction of the problem above has singularity, which brings difficulty in bounding the eigenvalues. For this problem, we develop new method to deal with the singularity. Moreover, the Lehmann-Goerisch theorem is applied to produce high precision eigenvalue bounds.

1. Introduction

To solve the eigenvalue problem (1), we consider the weak form: find $u \in H_0^1(\Omega)$

$$\lambda \in R, \text{ s.t. } (\nabla u, \nabla v) + (\sigma u, v) = \lambda(u, v), \forall v \in H_0^1(\Omega)\,, \tag{2}$$

where $H_0^1(\Omega)$ is a kind of Sobolev function space; $(\cdot,\cdot)$ is the inner product in $L_2(\Omega)$ or $L_2(\Omega)^2$. Let's denote the eigenvalues of (2) by $\lambda_1 \le \lambda_2 \le \cdots$.

The problem 2 can be solved approximately by using finite element method. Let $\mathcal{T}^h$ be a triangulation of domain Ω. The piecewise-continuous linear finite element space $V^h \subset H_0^1(\Omega)$, which has the hat function as its basis function, is adopted here as the approximation space. Suppose that $\dim(V^h) = n$. The Ritz-Galerkin method solves the variation problem (2) in V^h: Find $\lambda^h \in \mathbb{R}$ and $u_h \in V^h$ s.t.

$$(\nabla u_h, \nabla v_h) + (\sigma u_h, v_h) = \lambda^h\,(u_h, v_h), \quad \forall v_h \in V^h\,. \tag{3}$$

The eigenvalue problem (3) has finite eigen-pairs, which we denote by $\{\lambda_i^h, u_i^h\}_{i=1}^n$ and assume $\lambda_1^h \le \lambda_2^h \le \cdots \le \lambda_n^h$ and $(u_i^h, u_j^h) = \delta_{i,j}$.

The high precision bounds for the nth leading eigenvalues $\{\lambda_i\}_{i=1}^n$ are obtained in three steps.

Step 1: the base eigenvalue problem $-\Delta u = \lambda u$, that is, $\sigma = 0$, is solved approximately by finite element method as in (3), and an error estimation for the approximate eigenvalues is given in [2].

Step 2: the eigenvalue bounds for general elliptic operator in consideration is obtained by applying the homotopy method [2], which estimates the eigenvalue variation in transforming the base problem $-\Delta u = \lambda u$ to the one wanted. If the domain is convex, this step can be simplified by extending the result of [2].

Step 3: the Lehmann-Goerish's theorem [3,4] is applied to sharpen the bounds along with proper selection of base function to approximate the eigenfunction. To deal with the domain of free shape, the singular base function corresponding to the singular part of eigenfunction, and Bezier patch over triangulation of domain are used.

The structure of this paper is as follows. In section 2, 3, 4, we display the main theorems needed in our frame to estimate eigenvalues. In section 5, we illustrate an examples to demonstrate the efficiency of our proposed method.

2. Explicit upper bound and lower bound for eigenvalues in case $\sigma = 0$

First, we construct an explicit error estimation for the FEM solution of Poisson's equation, which is the key part of the algorithm to bound the Laplacian eigenvalues. We consider Poisson's equation associated with homogeneous Dirichlet boundary condition: for $f \in L_2(\Omega)$, find u such that,

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega\,. \tag{4}$$

The weak formulation of above problem is to find u in $H_0^1(\Omega)$ such that,

$$(\nabla u, \nabla v) = (f, v) \text{ for } v \in H_0^1(\Omega)\,. \tag{5}$$

The FEM solution $u_h \in V^h$ is given by solving above weak formulation in V^h:

$$(\nabla u_h, \nabla v_h) = (f, v_h) \text{ for } v_h \in V^h\,. \tag{6}$$

The selection of $H_0^1(\Omega)$ and V^h in §1, along with Lax-Milgram's theorem, assures the existence and uniqueness of solution u and u_h.

The classical error estimation theory gives the *a priori* estimation for FEM solution in a qualitative way:

$$\|u - u_h\|_{H^1} \le Ch^\alpha \|f\|_{L_2},$$

where the constant C, independent from f, is bounded but usually unknown; the exponent α is the convergence order. In case of homogeneous Dirichlet or Neumann boundary condition over convex domain, we know that the solution u of (5) belongs to $H^2(\Omega)$, for which we call H^2-regularity (see, e.g., [5]), and the order α can be one. However, such a regularity can not be expected for a non-convex domain or the mixed boundary condition, which brings difficulty to the FEM error estimation.

Next, by applying the technique of *hypercircle method* from Prager-Synge's theorem[9], we will develop concrete *a priori* error estimation in the form as follows,

$$|u-u_h|_{H^1} \le M_h\|f\|_{L_2}, \quad \|u-u_h\|_{L_2} \le M_h^2\|f\|_{L_2}, \quad (7)$$

where M_h is a computable quantity depending only on the triangulation of domain.

First, let's review several classical finite element spaces.

- Piecewise constant function space X^h:

  $X^h := \{v \in L_2(\Omega) | v \text{ is constant on each element of } \mathcal{T}^h\}$

- Lowest order Raviart-Thomas mixed FEM space W^h:

  $$W^h := \{\, p_h \in H(\mathrm{div},\Omega) \mid p_h = (a_K + c_K x, b_K + c_K y) \text{ in } K \in \mathcal{T}^h\}\,,$$

  where a_K, b_K, c_K are constants on element K and $H(\mathrm{div},\Omega)$ is defined by

  $$H(\mathrm{div},\Omega) = \{q \in (L_2(\Omega))^2 \mid \mathrm{div}\, q \in L_2(\Omega)\}\,.$$

- Subspace $W^h_{f_h}$ of W^h corresponding to $f_h \in X^h$:

  $$W^h_{f_h} := \{\, p_h \in W^h \mid \mathrm{div}\, p_h + f_h = 0 \text{ on each } K \in \mathcal{T}^h\}\,.$$

The classical analysis declares $\mathrm{div}(W^h) = X^h$ (c.f. Chapter IV.1 of [8]). Let us introduce the L_2-projection $\pi_{0,h}$: $L_2(\Omega) \to X^h$: for $v \in L_2(\Omega)$, $\pi_{0,h}v \in X^h$ satisfies,

$$(v - \pi_{0,h}v, v_h) = 0, \quad \forall v_h \in X^h. \quad (8)$$

The computable interpolation error estimation for $\pi_{0,h}$ has been well investigated, for example, c.f., Liu and Kikuchi [10], which reads,

$$\|v - \pi_{0,h}v\|_{L_2} \le C_0 h |v|_{H^1} \text{ for } v \in H^1(\Omega), \quad (9)$$

where h is the mesh size; constants C_0 has concreate values.

We introduce a computable quantity κ_h for purpose of developing the *a priori* estimation.

$$\kappa_h := \max_{f_h \in X^h\setminus\{0\}} \min_{v_h\in V^h} \min_{p_h \in W^h_{f_h}} \|p_h - \nabla v_h\|_{L_2}/\|f_h\|_{L_2} \quad (10)$$

In [2], we developed the computable *a priori* estimation under the above preparation.

Theorem 2.1 (*A priori* error estimation) *For any $f \in L_2(\Omega)$, let $u \in H^1_0(\Omega)$ and $u_h \in V^h$ be the solutions of variational problems (5) and (6), respectively. Let $M_h := \sqrt{C_0^2h^2 + \kappa_h{}^2}$, where C_0 is the constant appearing in (9). Then, we have,*

$$|u-u_h|_{H^1} \le M_h\|f\|_{L_2}, \quad \|u-u_h\|_{L_2} \le M_h^2\|f\|_{L_2}\,. \quad (11)$$

By using the quantity M_h, in [2], we obtained an explicit lower and upper bound for eigenvalues of L in case of $\sigma = 0$.

Theorem 2.2 ([2]) *Suppose $\lambda_k M_h^2 < 1$. Then the lower bound for λ_k is given by*

$$\lambda_k \ge \lambda_k^h/(1 + M_h^2\lambda_k^h). \quad (1 \le k \le n). \quad (12)$$

3. Homotopy method

The homotopy method is developed independently in [12] and [11]. Suppose A_0 and A_1 are two differential operators associated with eigenvalues as $\lambda_1^0 \le \lambda_2^0 \le \cdots$, and $\lambda_1^1 \le \lambda_2^1 \le \cdots$, respectively. Usually, the eigenvalues $\{\lambda_k^0\}$'s are known, which we call by *base problem*, and $\{\lambda_k^1\}$ are the one to estimate. The main idea of *homotopy method* is to introduce an intermediate operator A_s as followed:

$$A_s := (1-s)A_0 + sA_1.$$

The eigenvalues of A_s, denoted by $\{\lambda_k^s\}$, are supposed to satisfy

$$\lambda_k^0 \le \lambda_k^{s_1} \le \lambda_k^{s_2} \le \lambda_k^1, \quad 0 \le s_1 \le s_2 \le 1.$$

By using such intermediate operator A_s, one can obtain a lower bound ρ for certain λ_N, $\lambda_{N-1} < \rho \le \lambda_N$ [11]. Such a bound ρ may be rough, but it can be sharpened by futher applying Lehmann-Goerisch's theorem.

To apply the homotopy method to our problem, we introduce a constant σ_0 such that $\sigma_0 + \sigma(x) > 0$ on domain Ω. Thus, a homotopy can be created between the following two operators:

$$A_0u := -\Delta u, \quad A_1u := -\Delta u + (\sigma + \sigma_0)u$$

The base problem for $A_0u = -\Delta u$ is already solved by the estimation in (12). Then we can apply the homotopy method to bound eigenvalues of A_1. As the eigenvalues of $Lu = -\Delta u + \sigma u$ is just a shift of the ones of A_1 by $-\sigma_0$, the eigenvalues bounds of L are obtained imediately.

4. Improve the precision of egienvalue bounds

The eigenvalue bound from finite element method and the homotopy method are usually too rough. Here, we introduce Lehmann's method [14], which takes use of *a priori* rough lower bound for eigenvalues to sharpen the eigenvalue bounds. Such a method has an extention by F. Goeriesh for applicability, see, [13]. We display Lehmann's method in the following theorem, for which, a concise proof can be found in M. Plum [1].

Theorem 4.1 *Let L be self-adjoint operator defined over Hilbert space V with norm $\langle , \rangle$. Let the eigenvalues of $Lu = \lambda u$ be $\{\lambda_1, \lambda_2, \cdots\}$ in an increasing order. Take m linearly independent function $v_1, \cdots, v_m$ from V. Define $m \times m$ matrix A_1, A_2, A_3:*

$$A_1(i,j) := \langle Lv_i, v_j \rangle, A_2(i,j) := \langle v_i, v_j \rangle, A_3(i,j) := \langle Lv_i, Lv_j \rangle.$$

Denote by Λ_m the upper bound of λ_m obtained by Rayleigh-Ritz's method, that is, the maximum eigenvalue of $A_1 x = \lambda A_2 x$. Take a quantity ν such that,

$$\Lambda_m < \nu \le \lambda_{m+1} \ . \tag{13}$$

Define matrices B_1, B_2 by

$$B_1 := A_1 - \nu A_2, \quad B_2 := A_3 - 2\nu A_1 + \nu^2 A_2 \ .$$

Thus, $(-B_1)$ and B_2 must be positive definite matrices. Let the eigenvalues of eigen-problem $B_1 x = \mu B_2 x$ be $\mu_1 \le \mu_2 \le \cdots \le \mu_m < 0$. Then lower bound for eigenvalue problem $Lu = \lambda u$ is given by

$$\lambda_{m+1-k} \ge \nu + \frac{1}{\mu_k} \quad (k = 1, \cdots, m). \tag{14}$$

Remark 4.2 *The test functions v_i's will be well designed to approximate the exact eigenfunction accurately. Once v_i's are fixed, the lower bound given by (14) is monotonically increasing on variable ν. However, the selection of ν is not so sensible. Even rough lower bound ν of λ_{m+1} can provide precise lower bounds through (14).*

The lower bound from Homotopy method is a good choice of ν in (13). To find proper trial functions $\{v_i\}$, we adopt C^1 continuous Bernstein polynomial function over triangle element or rectangle element as basis function. Also, to approximate the eigenfunctions which is singular around the re-entrant corner, we will introduce trial functions having a singular part.

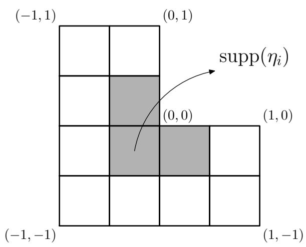


Figure 1: Uniform rectangle mesh of L-shaped domain and support of η_i ($h = 1/3$)

Let's take the uniform rectangular mesh $\mathcal{T}^h$ of L-shaped domain as an example, c.f., Figure 4. Define trial function $v \in C^1(\Omega)$ by

$$v := \sum_{K \in \mathcal{T}^h} P_K^N(x,y) + b_1 \eta_1 + b_2 \eta_2 \tag{15}$$

where $P_K^N(x,y)$ is the Bernstein polynomial of degree N over K; $\eta_k' s$ $(k = 1, 2)$ are singular functions with support only on elements neighbour to the corner,

$$\eta_k(r,\theta) = r^{2k/3} \sin \frac{2k}{3}(\theta + \pi/2)\,\Psi \quad (k = 1, 2).$$

Here Ψ is a C^1 continuous cut-off function that makes η_k vanish on the elements not connected to the re-entrant corner:

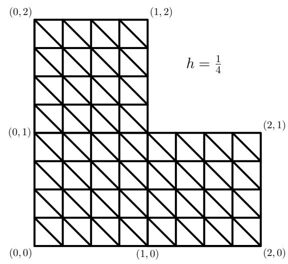
$$\Psi := (1-x^2/h^2)^2(1-y^2/h^2)^2 \text{ for } |x|, |y| \le h; \Psi := 0, \text{ otherwise.}$$

where h is the mesh size. The C^1 continuous condition of v and boundary condition will bring constraint conditions to the construction of P_K^N. We point out that $\eta_1 \notin H^2(\Omega)$, but $\Delta\eta_1 \in L_2(\Omega)$.

Thus, the m test functions $\{v_i\}_{i=1,\cdots,m}$ in Theorem 4.1 can be taken as the eigenfunctions corresponding to the leading mth eigenvalues obtained by solving variational eigenvalue problem with trial functions of the form (15).

5. Computation eaxmaple

The eigenvalue problem with homogeneous Dirichlet boundary condition over domain $\Omega = [0,2] \times [0,2] \setminus [1,2] \times [1,2]$ is considered here. The quantities needed by the estimation of projection P_h, such that, κ_h, $C_0 h$ and M_h, are displayed in Table 1. In this case, M_h tends to zero in order about 0.7, that is, $M_h = O(h^{0.7})$.



h	κ_h	$C_0 h$	M_h	order of κ_h
1/4	0.1466	0.080	0.1668	-
1/8	0.0882	0.040	0.0968	0.73
1/16	0.0538	0.020	0.0574	0.71
1/32	0.0332	0.010	0.0348	0.70

Table 1: Uniform mesh and values of κ_h

To apply Lehmann's method, we take $\nu := 39 < \lambda_6$. Thus we can obtain more precise bound for $\lambda_1, \cdots, \lambda_5$, where the polynomial P_K^N's degree is selected to be $N = 8$, $N = 10$ and singular basis introduced in §**??** are used. The result is displayed in Table 3. As numerical quadrature is

λ_i	Lower bound	Approx.	Upper bound	RelErr
1	9.5585	9.63972	9.6699	0.012
2	14.950	15.1970	15.225	0.018
3	19.326	19.7392	19.787	0.024
4	28.605	29.5215	29.626	0.035
5	30.866	31.9126	32.058	0.038
6	39.687	41.4745	41.680	0.049

Table 2: Eigenvalue evaluation based on (12) ($h = 1/32$)

λ_i	N=6	N=10
1	9.639_{55}^{73}	9.6397_{17}^{24}
2	15.1972_{30}^{53}	15.197251_{75}^{93}
3	19.7329208_{65}^{81}	19.7392088021_{25}^{80}
4	29.521_{34}^{49}	29.5214_{77}^{82}
5	31.912_{18}^{64}	31.9126_{21}^{36}

Table 3: High precision eigenvalues bounds

used to calculate the inner product of singular function, for the moment, we can not guarantee the numerical results of Table 3.

The general differential operator L, that is, $\sigma \neq 0$, for which the homotopy method is needed, will be displayed in the talk on the conference.

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