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On the Interpolation Constants over Triangular Elements

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Abstract—We present remarkable formulas which give sharp upper bounds for the interpolation constants on the triangles. These constants play an important role in the interpolation theory, in an a priori error estimation in the finite element analysis and in many other areas. The proposed formulas provide better upper bounds than the former ones. Moreover, they are convenient in practical calculations. In the proof of the formulas, we employ numerical verification method.

1. Introduction and Main Results

Let T be the triangle in $\mathbb{R}^2$ and $V^{1,1}(T), V^{1,2}(T), V^2(T)$ be the function spaces defined by

$$V^{1,1}(T) = \left\{\varphi \in H^1(T) \,\middle|\, \iint_T \varphi\, dxdy = 0\right\},$$
$$V^{1,2}(T) = \left\{\varphi \in H^1(T) \,\middle|\, \int_{\gamma_k} \varphi\, ds = 0, \quad k = 1,2,3\right\},$$
$$V^2(T) = \left\{\varphi \in H^2(T) \,\middle|\, \varphi(p_k) = 0, \quad k = 1,2,3\right\},$$

where p_1, p_2, p_3 and $\gamma_1, \gamma_2, \gamma_3$ are the vertices and sides of T respectively. Then, it is known that the following constants $C_1(T), C_2(T), C_3(T)$ and $C_4(T)$ exist:

$$C_1(T) = \sup_{\varphi \in V^{1,1}(T)\backslash 0} \frac{\|\varphi\|_{L^2(T)}}{\|\nabla\varphi\|_{L^2(T)}},$$
$$C_2(T) = \sup_{\varphi \in V^{1,2}(T)\backslash 0} \frac{\|\varphi\|_{L^2(T)}}{\|\nabla\varphi\|_{L^2(T)}},$$
$$C_3(T) = \sup_{\varphi \in V^2(T)\backslash 0} \frac{\|\varphi\|_{L^2(T)}}{|\varphi|_{H^2(T)}},$$
$$C_4(T) = \sup_{\varphi \in V^2(T)\backslash 0} \frac{\|\nabla\varphi\|_{L^2(T)}}{|\varphi|_{H^2(T)}},$$

where $|\varphi|_{H^2(T)}$ is a H^2 semi-norm of φ defined by

$$|\varphi|^2_{H^2(T)} = \|\varphi_{xx}\|^2_{L^2(T)} + 2\|\varphi_{xy}\|^2_{L^2(T)} + \|\varphi_{yy}\|^2_{L^2(T)}.$$

A lot of work has been done on the upper bounds of these constants. For example, [11, 6, 5] for constant $C_1(T)$, [3, 7] for $C_2(T)$, [2] for $C_3(T)$ and [12, 2, 3, 8, 10, 1, 9, 5, 7] for $C_4(T)$.

For these constants, we have obtained the formulas which give sharp upper bound of $C_j(T)$ as

$$C_j(T) < K_j(T), \qquad j = 1,2,3,4.$$

Concrete form of the formulas $K_j(T)$ are the following[4]:

$$K_1(T) = \sqrt{\frac{A^2+B^2+C^2}{28} - \frac{S^4}{A^2B^2C^2}},$$
$$K_2(T) = \sqrt{\frac{A^2+B^2+C^2}{54} - \frac{S^4}{2A^2B^2C^2}},$$
$$K_3(T) = \sqrt{\frac{A^2B^2+B^2C^2+C^2A^2}{83} - \frac{1}{24}\left(\frac{A^2B^2C^2}{A^2+B^2+C^2} + S^2\right)},$$
$$K_4(T) = \sqrt{\frac{A^2B^2C^2}{16S^2} - \frac{A^2+B^2+C^2}{30} - \frac{S^2}{5}\left(\frac{1}{A^2} + \frac{1}{B^2} + \frac{1}{C^2}\right)},$$

where A, B, C are the edge length of triangle T and S is the area of T.

We proved these formulas by the aid of numerical verification method and asymptotic analysis.

2. Graphs

Since we don't have enough space here, we will only show the graph of $\widetilde{C}_j(T), K_j$ and $K_j(T) - \widetilde{C}_j(T)$ in the next page. In the graphs, $\widetilde{C}_j(T)$ are approximate solutions of $C_j(T)$ and T is a triangle whose vertices are $(0,0), (1,0)$ and (a,b).

Details of the proof will be explained in the talk.

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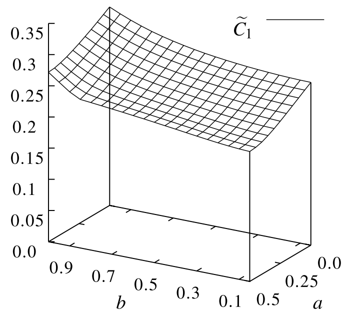

Figure 1: $\widetilde{C}_1(T)$

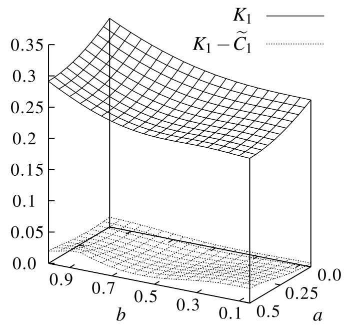

Figure 2: K_1 and $K_1(T) - \widetilde{C}_1(T)$

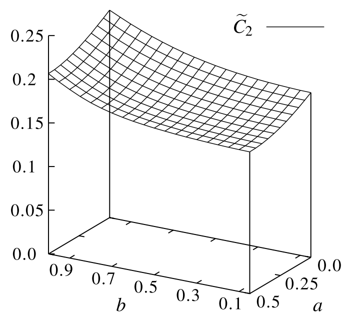

Figure 3: $\widetilde{C}_2(T)$

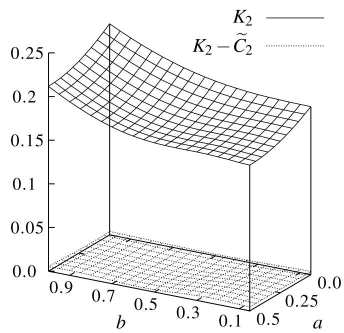

Figure 4: K_2 and $K_2(T) - \widetilde{C}_2(T)$

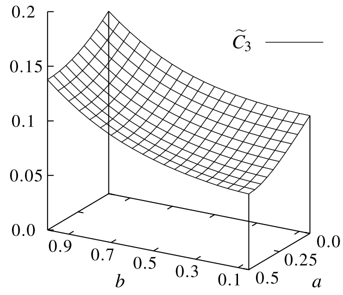

Figure 5: $\widetilde{C}_3(T)$

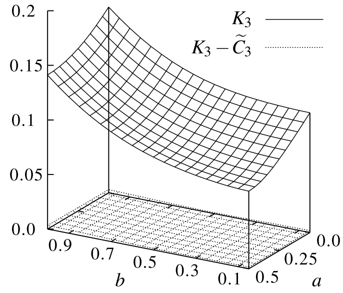

Figure 6: K_3 and $K_3(T) - \widetilde{C}_3(T)$

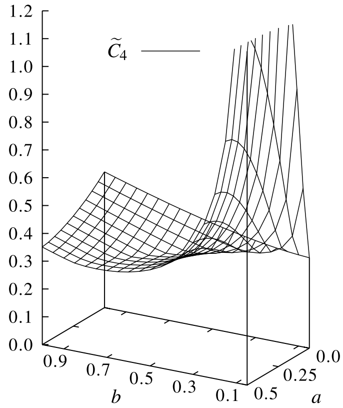

Figure 7: $\widetilde{C}_4(T)$

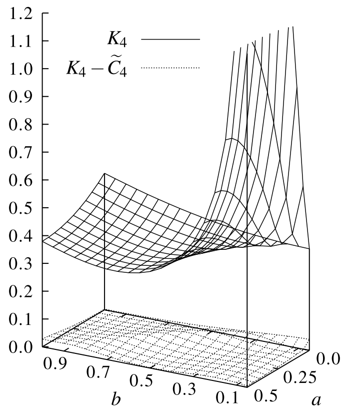

Figure 8: K_4 and $K_4(T) - \widetilde{C}_4(T)$

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