IEICE Proceeding Series

Bifurcation analysis of Kerr optical frequency comb generation

Irina Balakireva, Rémi Henriet, Aurélien Coillet, Laurent Larger, Yanne K. Chembo

Vol. 1 pp. 779-782
Publication Date: 2014/03/17
Online ISSN: 2188-5079

2012 International Symposium on Nonlinear Theory and its Applications
NOLTA2012, Palma, Majorca, Spain, October 22-26, 2012

Bifurcation analysis of Kerr optical frequency comb generation

Irina Balakireva, Rémi Henriet, Aurélien Coillet, Laurent Larger and Yanne K. Chembo

FEMTO-ST Institute [CNRS UMR6174], Optics Department,
16 Route de Gray, 25030 Besançon cedex, France
Email: yanne.chembo@femto-st.fr

Abstract—We perform a detailed bifurcation analysis of Kerr optical frequency combs generated with monolithic whispering gallery mode resonators. We use a multimode dynamics approach to investigate the stability of Kerr combs excited by an external laser pump. Our results enable to understand accurately the process of Kerr comb formation through a nonlinear bifurcation analysis.

1. Introduction

Optical frequency combs are a set of equidistant spectral lines in the Fourier spectrum. This feature is useful in many applications such as sensing, optical engineering, metrology, or microwave generation. It has been shown recently that optical frequency comb could be generated using the whispering gallery mode (WGMs) of a monolithic resonator. The essential advantages of this system would be intrinsic simplicity, small size, low power consumption and higher performances in terms of phase noise.

In WGM optical frequency comb generators, the dielectric micro-resonator is shaped as a cylinder, a sphere or a torus whose principal dimension may range from few tens of micrometers to few millimeters. Provided that the bulk material is low-loss and the resonator is properly prepared (sub-nanometer surface irregularities), the light is trapped within the eigenmodes of the micro-cavity, which are usually referred to as whispering gallery modes. Their free-spectral range may vary from few GHz to few THz depending on the resonator's radius, and their quality factor Q can be exceptionally high, greater than 10^{11}. In these resonators, the small volume of confinement, high photon density and long photon storage time (proportional to the quality factor Q) induce a very strong light-matter interaction. Depending on the dielectric material, this strong coupling can generate a highly efficient Four-Wave Mixing (FWM), where two pump photons are transformed into two side-bands photons through the Kerr nonlinearity. Provided that the pump is powerful enough, an optical frequency comb, sometimes referred to as Kerr comb, is generated through a cascaded creation of such side-bands photons, resulting from the interaction of the pump and the WGMs via the Kerr nonlinearity (see [1, 2, 3] and references therein).

However, the theory of comb generation using WGMs is not complete and several phenomena are still unexplained. This is due to the fact that this spatiotemporal system is strongly nonlinear, multidimensional, and involves several parameters that are coupled in a very complex, nontrivial fashion. In this paper, we describe a multimode dynamics approach to study this system, with an emphasis on the role played by the nonlinearity, the Q factor of the resonator, and the cavity dispersion. In particular, we develop a bifurcation analysis showing how the comb unfolds as the main features of the pumping (power and detuning) are varied.

2. The system

The system basically consists in a symmetrically truncated spherical cavity, coupled to the angle-polished tips of two optical fibers, one for light injection and the other for photon extraction. Other configurations may be considered as well (fiber taper, waveguide or prism couplings for example), as they could also be investigated with the formalism hereafter developed.

The electric field in the cavity can be expanded according to the WGMs, following

$$\mathbf{E}(\mathbf{r}, t) = \sum_{\eta} \frac{1}{2}\mathcal{E}_{\eta}(t)e^{i\omega_{\eta}t}\boldsymbol{\Upsilon}_{\eta}(\mathbf{r}) + \frac{1}{2}\mathbf{E}_{ext}e^{i\Omega_0 t} + \text{c.c.}, \quad (1)$$

where η stands for the various modes under consideration, defined by a set $\boldsymbol{\Upsilon}_{\eta}(\mathbf{r})$ of orthonormal and vectorial eigenmodes of frequency ω_{η}, and by their time-varying amplitude $\mathcal{E}_{\eta}(t)$. The constant vector $\mathbf{E}_{ext}$ stands for the external laser excitation of frequency Ω_0 while "c.c." stands for the complex conjugate of all the preceding terms.

The explicit expression of the WGMs of a spherical cavity can be obtained analytically, and have been for long investigated within the frame of morphology dependent resonances. The unknowns are therefore only the modal amplitudes. We have used a hermitian projection technique to derive an explicit set of modal equations describing the dynamics of the electric field in the various WGMs. We can restrict ourselves to the quasi-degenerate limit for the modal equations parameters, as long as we are only interested in the dynamics of WGMs that are not too far from the pump frequency. In this case, if the field is normalized in such a way that $|\mathcal{A}_{\eta}|^2$ is equal to the number of photons in the mode η, the modal dynamics equations can be explicitly written as

$$\dot{\mathcal{A}}_\eta = -\frac{1}{2}\Delta\omega_0 \mathcal{A}_\eta - ig_0 \sum_{\alpha,\beta,\mu} \mathcal{A}_\alpha \mathcal{A}^*_\beta \mathcal{A}_\mu e^{i\varpi_{\alpha\beta\mu\eta}t} + \frac{1}{2}\Delta\omega_0 \mathcal{F}_\eta e^{i(\Omega_0-\omega_\eta)t} \quad (2)$$

where $\Delta\omega_0$ is the bandwidth of the loaded WGMs, g_0 is the nonlinear gain induced by the Kerr nonlinearity, $\mathcal{F}_\eta$ is the modal field injection from the outside, Ω_0 is the angular frequency of the laser pump excitation and $\varpi_{\alpha\beta\mu\eta} = \omega_\alpha - \omega_\beta + \omega_\mu - \omega_\eta$ is a frequency detuning induced by the four-wave mixing process. It is noteworthy that this detuning frequency also includes both the geometrical and material dispersion properties of the resonator, as it is equal to zero in the dispersionless limit.

3. Results

It is convenient to label the various eigenmodes with their reduced angular eigennumber $l = \ell - \ell_0$, where ℓ_0 is the angular number of the pumped mode. With this notation, the pumped mode corresponds to $l = 0$, while the sidemodes symmetrically expand as $\pm l$ around the central mode.

When we pump the cavity below threshold, only the mode $l = 0$ is excited. At threshold, the pair of side modes $\mathcal{A}_{\pm l}$ appears, and an important issue is to investigate the stability condition for the excitation of a given pair $\pm l$ of sidemodes. This study can be performed within the frame of so-called primary comb approximation. In fact, the primary comb is constituted with the oscillating sidemodes whose amplitude is exclusively due to the pump, while their phase may be affected by other sidemodes (which are themselves exclusively pump induced). In other words, all the photons that are in the sidemodes $\pm l$ originate from the pump through the photonic interaction $2\hbar\omega_0 \to \hbar\omega_l + \hbar\omega_{-l}$, while the pump is being depleted accordingly. However, some photons are fed back through the reverse interaction $\hbar\omega_l + \hbar\omega_{-l} \to 2\hbar\omega_0$. The advantage of this paradigm is that the excited modes can in turn be considered as secondary pumps exciting other modes on their own. The process cascades up and down to generate the total comb were any four modes fulfilling energy and angular momentum conservation requirements may *a priori* interact. This stability analysis has already been performed in ref. Yanne, and it was shown that the threshold number of photons in the pumped mode was:

$$|\mathcal{A}_0|^2_{th} = \frac{1}{2}\frac{\Delta\omega_0}{g_0}. \quad (3)$$

So when we pump the cavity above the first order threshold, we now have three modes: namely 0 and $\pm l$. From the Eq. (2), we can derive the dynamical equations for each of these three modes:

$$\dot{\mathcal{A}}_0 = -\frac{1}{2}\Delta\omega_0\mathcal{A}_0 - ig_0\Big\{|\mathcal{A}_0|^2\mathcal{A}_0 + 2|\mathcal{A}_{-l}|^2\mathcal{A}_0 + 2|\mathcal{A}_{+l}|^2\mathcal{A}_0 + 2\mathcal{A}_{-l}\mathcal{A}_{+l}\mathcal{A}^*_0 e^{-i\varpi_{\pm l}t}\Big\} + \frac{1}{2}\Delta\omega_0\mathcal{F}_0 e^{i\sigma t} \quad (4)$$

$$\dot{\mathcal{A}}_{\pm l} = -\frac{1}{2}\Delta\omega_{\pm l}\mathcal{A}_{\pm l} - ig_0\Big\{2|\mathcal{A}_0|^2\mathcal{A}_{\pm l} + 2|\mathcal{A}_{\mp l}|^2\mathcal{A}_{\pm l} + |\mathcal{A}_{\pm l}|^2\mathcal{A}_{\pm l} + \mathcal{A}^2_0\mathcal{A}^*_{\mp l}e^{i\varpi_{\pm l}t}\Big\} \quad (5)$$

where $\varpi_{\pm l} = 2\omega_0 - \omega_{\mp l} - \omega_{\pm l}$ is the modal detuning. We can remove the explicit time dependence in this equation by introduction the transformation

$$\mathcal{B}_0 = \mathcal{A}_0 e^{-i\sigma t} \quad (6)$$

$$\mathcal{B}_{\pm l} = \mathcal{A}_{\pm l} e^{-i(\sigma+\frac{1}{2}\varpi_{\pm l})t} \quad (7)$$

obeying

$$\dot{\mathcal{B}}_0 = -i\sigma\mathcal{B}_0 - \frac{1}{2}\Delta\omega_0\mathcal{B}_0 - ig_0\Big\{|\mathcal{B}_0|^2\mathcal{B}_0 + 2|\mathcal{B}_{-l}|^2\mathcal{B}_0 + 2|\mathcal{B}_{+l}|^2\mathcal{B}_0 + 2\mathcal{B}_{-l}\mathcal{B}_{+l}\mathcal{B}^*_0\Big\} + \frac{1}{2}\Delta\omega_0\mathcal{F}_0 \quad (8)$$

and

$$\dot{\mathcal{B}}_{\pm l} = -i(\sigma + \frac{1}{2}\varpi_{\pm l})\mathcal{B}_{\pm l} - \frac{1}{2}\Delta\omega_{\pm l}\mathcal{B}_{\pm l} - ig_0\Big\{|\mathcal{B}_{\pm l}|^2\mathcal{B}_{\pm l} + 2|\mathcal{B}_{\mp l}|^2\mathcal{B}_{\pm l} + 2|\mathcal{B}_0|^2\mathcal{B}_{\pm l} + \mathcal{B}^2_0\mathcal{B}^*_{\mp l}\Big\} \quad (9)$$

Now we can rewrite $\mathcal{B}$ as $\mathcal{B}_0 = B_0\exp[i\phi_0]$ and also $\mathcal{B}_{\pm l} = B_{\pm l}\exp[i\phi_{\pm l}]$. Hence, we can suppose that the sidemodes have the same amplitudes $B_{+l} = B_{-l}$, but different phases $\phi_{+l} \neq \phi_{-l}$. Finally, using the stationarity relationships $\dot{\mathcal{B}}_\eta = 0$, we can get from Eq. (8) and Eq. (9) the equation

$$|\mathcal{F}_0|^2 = \frac{4g_0^2}{\Delta\omega_0^2}|\mathrm{B}_0|^6 + \frac{8\sigma g_0}{\Delta\omega_0^2}|\mathrm{B}_0|^4 + \Big\{[1 + \frac{4\sigma^2}{\Delta\omega_0^2}] - \frac{8g_0}{\Delta\omega_0^2}(2\sigma + \varpi_{\pm l})|\mathrm{B}_{\pm l}|^2 - \frac{96g_0^2}{\Delta\omega_0^2}|\mathrm{B}_{\pm l}|^4\Big\}|\mathrm{B}_0|^2 + \Big\{\frac{4}{\Delta\omega_0^2}[\Delta\omega_{\pm l}\Delta\omega_0 - 4\sigma^2 - 2\sigma\varpi_{\pm l}]|\mathrm{B}_{\pm l}|^2 - \frac{4g_0}{\Delta\omega_0^2}[28\sigma + 8\varpi_{\pm l}]|\mathrm{B}_{\pm l}|^4 - \frac{192g_0^2}{\Delta\omega_0^2}|\mathrm{B}_{\pm l}|^6\Big\} \quad (10)$$

ruling the the relationship between the external pump and the excited modes. From Eq. (9) we can also get:

$$\sin\phi = -\frac{\Delta\omega_{\pm l}}{2g_0|\mathrm{B}_0|^2} \quad (11)$$

and

$$\cos\phi = \frac{-(\sigma + \frac{1}{2}\varpi_{\pm l}) - 3g_0|\mathrm{B}_{\pm l}|^2 - 2g_0|\mathrm{B}_0|^2}{g_0|\mathrm{B}_0|^2} \quad (12)$$

with $\phi = \phi_{\pm l} + \phi_{\mp l} - 2\phi_0$. This relationships demonstrate that ϕ depends only on $\Delta\omega_{\pm l}$.

We can get the dependence between $|\mathsf{B}_0|$ and $|\mathsf{B}_{\pm l}|$ using Eq. (9):

$$36g_0^2|\mathsf{B}_{\pm l}|^4 + \Big\{24\sigma g_0 + 12\varpi_{\pm l}g_0 + 48g_0^2|\mathsf{B}_0|^2\Big\}|\mathsf{B}_{\pm l}|^2 + \Delta\omega_{\pm l}^2 + [2\sigma + \varpi_{\pm l}]^2 + 8g_0[2\sigma + \varpi_{\pm l}]|\mathsf{B}_0|^2 + 12g_0^2|\mathsf{B}_0|^4 = 0\,, \quad (13)$$

so that now we have a system of two equations with two unknowns. We can therefore calculate the value of the stationary variables $|\mathsf{B}_0|^2$, $|\mathsf{B}_{\pm l}|^2$ and also ϕ.

As we continue to increase the pump power, we will obsere the emergence of sub-harmonic comb components, which will appear after crossing the so-called second order threshold. For the sake of simplification, we will consider only five modes, namely 0, $\pm l$ and the two sub-harmonic modes $\pm l/2$.

In order to find the threshold leading to oscillation for a given pair of side modes $\mathcal{A}_{\pm l/2}$, a well-known technique is to investigate the linear stability of the trivial equilibrium $\mathcal{A}_{\pm l/2} = 0$. This equilibrium is perturbed with $\delta\mathcal{A}_{\pm l/2}$, and the threshold is defined by the set of parameters separating the values for which the perturbation decays to 0 (the trivial equilibrium is stable) of those where the perturbation diverges to infinity (onset of oscillations).

The sidemode perturbation equations for the secondary comb obey

$$\begin{aligned}
\delta\dot{\mathcal{A}}_{\pm l/2} &= \Big\{-\frac{1}{2}\Delta\omega_{\pm l/2} \\
&-ig_0\left[\Lambda_{\pm l/2}^{0,0,\pm l/2} + \Lambda_{\pm l/2}^{\pm l/2,0,0}\right]|\mathcal{A}_0|^2 \\
&-ig_0\left[\Lambda_{\pm l/2}^{\pm l/2,l,l} + \Lambda_{\pm l/2}^{l,l,\pm l/2}\right]|\mathcal{A}_{+l}|^2 \\
&-ig_0\left[\Lambda_{\pm l/2}^{-l,-l,\pm l/2} + \Lambda_{\pm l/2}^{\pm l/2,-l,-l}\right]|\mathcal{A}_{-l}|^2\Big\}\delta\mathcal{A}_{\pm l/2} \\
&-ig_0\Big\{\left[\Lambda_{\pm l/2}^{\mp l/2,0,\pm l} + \Lambda_{\pm l/2}^{\pm l,0,\mp l/2}\right] \\
&\mathcal{A}_0^*\mathcal{A}_{\pm l}e^{i[-\frac{1}{2}\varpi_{\mp l}+\frac{1}{2}(\varpi_{\pm B}-\varpi_{\mp B})]t} \\
&+\left[\Lambda_{\pm l/2}^{0,\mp l,\mp l/2} + \Lambda_{\pm l/2}^{\mp l/2,\mp l,0}\right] \\
&\mathcal{A}_0\mathcal{A}_{\mp l}^*e^{i[\frac{1}{2}\varpi_{\mp l}+\frac{1}{2}(\varpi_{\pm B}-\varpi_{\mp B})]t}\Big\}\delta\mathcal{A}_{\mp l/2} \\
&-ig_0\Big\{\left[\Lambda_{\pm l/2}^{0,\pm l/2,\pm l} + \Lambda_{\pm l/2}^{\pm l,\pm l/2,0}\right] \\
&\mathcal{A}_0\mathcal{A}_{\pm l}e^{i[\varpi_{\pm B}-\frac{1}{2}\varpi_{\pm l}]t}\Big\}\delta\mathcal{A}_{\pm l/2}^* \\
&-ig_0\Big\{\left[\Lambda_{\pm l/2}^{\mp l,\mp l/2,\pm l} + \Lambda_{\pm l/2}^{\pm l,\mp l/2,\mp l}\right] \\
&\mathcal{A}_{\mp l}\mathcal{A}_{\pm l}e^{i[\frac{1}{2}(\varpi_{\pm B}+\varpi_{\mp B})-\varpi_{\pm l}]t} \\
&+\Lambda_{\pm l/2}^{0,\mp l/2,0}\mathcal{A}_0^2e^{i[\frac{1}{2}(\varpi_{\pm B}+\varpi_{\mp B})]t}\Big\}\delta\mathcal{A}_{\mp l/2}^*
\end{aligned} \quad (14)$$

where

$$\varpi_{\pm B} = 2\omega_0 + \frac{1}{2}\omega_{\pm l} - \frac{1}{2}\omega_{\mp l} - 2\omega_{\pm l/2}$$

$$\varpi_{\pm l/2} = 2\omega_0 - \omega_{\mp l/2} - \omega_{\pm l/2} = \frac{1}{2}(\varpi_{\pm B} + \varpi_{\mp B}) \quad (15)$$

The intermodal coupling coefficients $\Lambda_\eta^{\alpha\beta\mu}$ converge to 1 in our case, and explicit time dependence can be removed by the introduction the variables

$$\begin{aligned}
\mathcal{B}_0 &= \mathcal{A}_0e^{-i\sigma t}, \\
\mathcal{B}_{\pm l} &= \mathcal{A}_{\pm l}e^{-i(\sigma+\frac{1}{2}\varpi_{\pm l})t} \\
\delta\mathcal{B}_{\pm l/2} &= \delta\mathcal{A}_{\pm l/2}e^{-i(\sigma+\frac{1}{2}\varpi_{\pm B})t}\,.
\end{aligned} \quad (16)$$

Hence, Eq. (14) can be rewritten as

$$\begin{aligned}
\delta\dot{\mathcal{B}}_{\pm l/2} &= \Big\{-\frac{1}{2}\Delta\omega_{\pm l/2} - i(\sigma + \frac{1}{2}\varpi_{\pm B}) \\
&-ig_0\left[\Lambda_{\pm l/2}^{0,0,\pm l/2} + \Lambda_{\pm l/2}^{\pm l/2,0,0}\right]|\mathcal{B}_0|^2 \\
&-ig_0\left[\Lambda_{\pm l/2}^{\pm l/2,l,l} + \Lambda_{\pm l/2}^{l,l,\pm l/2}\right]|\mathcal{B}_{+l}|^2 \\
&-ig_0\left[\Lambda_{\pm l/2}^{-l,-l,\pm l/2} + \Lambda_{\pm l/2}^{\pm l/2,-l,-l}\right]|\mathcal{B}_{-l}|^2\Big\}\delta\mathcal{B}_{\pm l/2} \\
&-\Big\{ig_0\left[\Lambda_{\pm l/2}^{0,\mp l,\mp l/2} + \Lambda_{\pm l/2}^{\mp l/2,\mp l,0}\right]\mathcal{B}_0\mathcal{B}_{\mp l}^* \\
&+ig_0\left[\Lambda_{\pm l/2}^{\mp l/2,0,\pm l} + \Lambda_{\pm l/2}^{\pm l,0,\mp l/2}\right]\mathcal{B}_0^*\mathcal{B}_{\pm l}\Big\}\delta\mathcal{B}_{\mp l/2} \\
&-\Big\{ig_0\left[\Lambda_{\pm l/2}^{0,\pm l/2,\pm l} + \Lambda_{\pm l/2}^{\pm l,\pm l/2,0}\right]\mathcal{B}_0\mathcal{B}_{\pm l}\Big\}\delta\mathcal{B}_{\pm l/2}^* \\
&-\Big\{ig_0\left[\Lambda_{\pm l/2}^{\mp l,\mp l/2,\pm l} + \Lambda_{\pm l/2}^{\pm l,\mp l/2,\mp l}\right]\mathcal{B}_{\mp l}\mathcal{B}_{\pm l} \\
&+ig_0\Lambda_{\pm l/2}^{0,\mp l/2,0}\mathcal{B}_0^2\Big\}\delta\mathcal{B}_{\mp l/2}^*
\end{aligned} \quad (17)$$

which can be simply rewritten as

$$\begin{aligned}
\delta\dot{\mathcal{B}}_{+l/2} &= \mathsf{M}_{+l/2}\delta\mathcal{B}_{+l/2} + \mathsf{N}_{+l/2}\delta\mathcal{B}_{-l/2} \\
&\quad+\mathsf{R}_{+l/2}\delta\mathcal{B}_{+l/2}^* + \mathsf{P}_{+l/2}\delta\mathcal{B}_{-l/2}^* \\
\delta\dot{\mathcal{B}}_{-l/2} &= \mathsf{M}_{-l/2}\delta\mathcal{B}_{-l/2} + \mathsf{N}_{-l/2}\delta\mathcal{B}_{+l/2} \\
&\quad+\mathsf{R}_{-l/2}\delta\mathcal{B}_{-l/2}^* + \mathsf{P}_{-l/2}\delta\mathcal{B}_{+l/2}^*
\end{aligned} \quad (18)$$

where the parameters $\mathsf{M}, \mathsf{R}, \mathsf{N}$ and P are complex. Since Equation (18) is complex-valued, it can be split in real and imaginary parts and after transform into the matricial form:

$$\begin{bmatrix} \Re[\delta\dot{\mathcal{B}}_{+l/2}] \\ \Re[\delta\dot{\mathcal{B}}_{-l/2}^*] \\ \Im[\delta\dot{\mathcal{B}}_{+l/2}] \\ \Im[\delta\dot{\mathcal{B}}_{-l/2}^*] \end{bmatrix} = [\mathbf{K}] \begin{bmatrix} \Re[\delta\mathcal{B}_{+l/2}] \\ \Re[\delta\mathcal{B}_{-l/2}^*] \\ \Im[\delta\mathcal{B}_{+l/2}] \\ \Im[\delta\mathcal{B}_{-l/2}^*] \end{bmatrix}. \quad (19)$$

The sidemodes $\pm l/2$ arise when at least one of the eigenvalues of the matrix $[\mathbf{K}]$ has a positive real value. This task is indeed mathematically heavy because we have a fourth order equation linear analysis problem, but exact analytical stability conditions may be derived using for example the Routh-Hurwitz criterion.

4. Conclusion

In conclusion, we have investigated the cascaded threshold behavior observed during the formation of Kerr optical frequency combs using monolithic whispering gallery mode oscillators. We have mainly focused our efforts on the secondary comb. Future work will focus on the analysis of the full bifurcation behavior, as well as on the comparison with experiments.

Acknowledgments

Y. K. C. acknowledges support from the European Research Council (ERC) through the StG project NextPhase. Authors also acknowledge support from the *Centre National de la Recherche Spatiale* (CNES) and from the *Région de Franche-Comté*.

References

[1] Y. K. Chembo and N. Yu, "Modal expansion approach to optical-frequency-comb generation with monolithic whispering-gallery-mode resonators," *Phys. Rev. A*, vol. 82, pp. 033801–1-18, 2010.

[2] Y. K. Chembo, D. V. Strekalov and N. Yu, "Spectrum and Dynamics of Optical Frequency Combs Generated with Monolithic Whispering Gallery Mode Resonators," *Phys. Rev. Lett.*, vol. 104, pp. 103902–1-4, 2010.

[3] Y. K. Chembo and N. Yu, "On the generation of octave-spanning optical frequency combs using monolithic whispering-gallery-mode microresonators," *Opt. Lett.*, vol. 35, pp. 2696–2698, 2010.