IEICE Proceeding Series

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Vol. 1 pp. 763-766
Publication Date: 2014/03/17
Online ISSN: 2188-5079

2012 International Symposium on Nonlinear Theory and its Applications
NOLTA2012, Palma, Majorca, Spain, October 22-26, 2012

Delay reduction in networks of coupled dynamical systems

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Abstract—We consider networks of coupled dynamical systems with delayed interactions and discuss the possibilities of delay reductions for arbitrary coupling topologies. Using appropriate timeshift transformations, the number of interaction delays can always be reduced to at most the dimension of the cycle space of the underlying graph. For instance, in a unidirectional ring we can reduce the number of different delays to one while the roundtrip delay time is preserved. More generally, the roundtrips along a set of fundamental cycles act as an important factor in determining the dynamical behavior.

1. Introduction

Propagation and processing time delays play an important role in many networks of coupled dynamical systems. As soon as the time delay is comparable with other characteristic time scales in the system, it may alter the dynamics significantly. For instance, the propagation of light between interacting lasers [22, 6, 10] causes delays, despite of the very fast propagation velocity of the light. The reason is that the internal timescales of the laser are small as well, so that a few centimeters of propagation distance may cause significant time delays. In neuronal networks [15, 19] delays occur due to the finite propagation times along the axons or to reaction times at chemical synapses.

In many cases the delays can be considered not as an undesirable feature of a system, which makes its analysis more complicated, but rather as an important ingredient allowing to produce desirable functionality [7]. For example, it is used for information processing [1], chaos-based communication [2], fast physical random number generators [16], pattern generation [21, 26], neural processing of temporal information [4], etc.

The dynamics of networks of interacting systems with various interaction delays can often be described by equations of the form

$$\dot{x}_j(t) = f_j\left(x_j(t), \left(x_k\left(t-\tau_{jk}\right)\right)_{k\in P_j}\right), \qquad (1)$$

where $x_j(t)$, $j = 1,\ldots,N$, denotes the dynamical state of a node j and P_j is the set of its predecessors. That is, for each $k \in P_j$, there exists a link $k \to j$ in the network. The corresponding connection delay is denoted by τ_{jk} and usually occurs due to the signal propagation time from a node k to the node j or the processing time of the incoming signal in node j. We assume that the network is connected and hence, the number L of links is larger than $N-1$.

For a network of N systems, up to N^2 different time delays may occur in (1). By allowing several connections from one system to another, the number of delays can increase even more. This creates immense challenges for the analysis as well as for the numerical simulation of the system, since every single delay may alter the system's properties and dynamics significantly [9, 11, 8, 13, 20, 23, 24, 25]. The interaction of several delays is even more complicated [10, 21, 26, 12, 14] and not fully understood so far.

In [17] we show that the number of different delays can be reduced. More precisely, any connected network possesses a characteristic number of delays which is essential for describing the dynamics. This number of essential delays equals the cycle space dimension $C = L-(N-1)$ of the underlying graph and is generically smaller than the number of distinct τ_{jk}. We show that the essential delays correspond to generalized roundtrip times along fundamental semicycles in the network. As a consequence, networks which have the same local dynamics and the same set of essential delays can be considered as equivalent from the dynamical point of view.

2. Delay transformation on a unidirectional ring

As a simple illustration, let us consider a unidirectional ring of identical systems

$$\dot{x}_j(t) = f\left(x_j(t), x_{j+1}\left(t-\tau_j\right)\right) \qquad (2)$$

with inhomogeneous coupling delays τ_j. This system is equivalent to a homogeneous ring where all time delays are equal to the N-th part of the roundtrip $\mathbf{rt} = \sum \tau_j$ [5, 20]. To prove this, we consider a componentwise timeshift, which, for a solution $x(t)$, $t \in \mathbb{R}$, may be written as

$$y_j(t) = x_j\left(t+\eta_j\right), \quad 1 \le j \le N. \qquad (3)$$

This means, each node is considered in its own time $t_j = t+\eta_j$ and the shape of the solution $x(t)$ is merely shifted componentwise by this transformation [Fig. 1]. Simple differentiation of the variables $y_j(t)$ shows, that the transformed system satisfies the equations

$$\dot{y}_j(t) = f\left(y_j(t), y_{j+1}\left(t-\tau_j+\eta_j-\eta_{j+1}\right)\right),$$

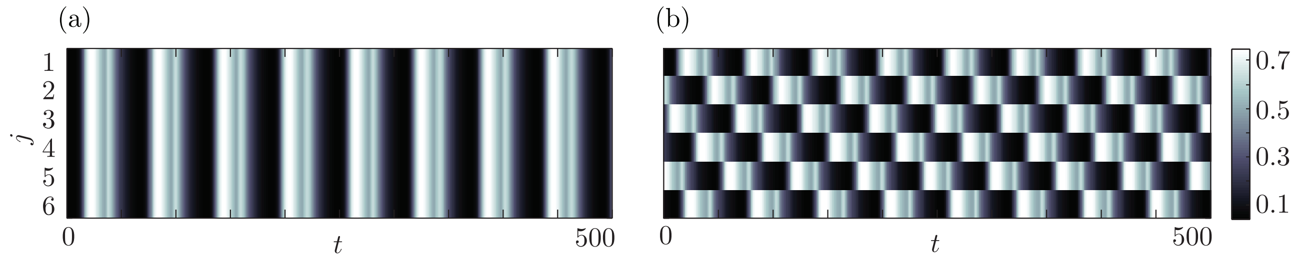

Figure 1: Illustration of the timeshift transformation (3) for a unidirectional ring (2) of Mackey-Glass elements $x_j(t)$, $j = 1, ..., N = 6$, as in [6]. The coloring of the j-th row corresponds to the state of the j-th node. (a) A synchronous periodic solution $x(t) = (x_1(t), ..., x_6(t))$ in the case of homogeneous connection delays $\tau_j \equiv \tau$. (b) The transformed solution $y(t) = (x_1(t + \eta_1), ..., x_6(t + \eta_6))$ with componentwise timeshifts η_j such that $\tilde{\tau}_j = \tau_j + \eta_{j+1} - \eta_j = 0$, for $j = 1, ..., 5$ and $\tilde{\tau}_6 = \tau_6 + \eta_1 - \eta_6 = 6\tau$.

where $\eta_{N+1} = \eta_1$. The only difference between (2) and (1) consists in the different connection delay times $\tilde{\tau}_j = \tau_j + \eta_{j+1} - \eta_j$. It is possible to choose shifts η_j such that all new delays become the same $\tau = \frac{1}{N}\mathbf{rt}$ while their sum along the ring, the roundtrip time $\mathbf{rt} = N\tau$, is preserved. Indeed, for any choice of the timeshifts η_j we have

$$\sum_j \tau_j + \eta_{j+1} - \eta_j = \sum_j \tau_j + \sum_j \eta_{j+1} - \sum_j \eta_j = \sum_j \tau_j.$$

This means the roundtrip $\mathbf{rt}$ is preserved under the transformation (3). This idea was applied in [3, 20] to reveal the hidden $\mathbb{Z}_N$-symmetry in systems of the form (2) and to study analytically its solutions.

3. Delay transformation for a general system

When the componentwise timeshift (3) is applied to systems with a more general coupling structure as described by equation (1), we obtain the transformed system

$$\dot{y}_j(t) = f_j\left(y_j(t), \left(y_k\left(t - \tilde{\tau}_{jk}\right)\right)_{k \in P_j}\right). \qquad (4)$$

Again, the difference to (1) only consists in the new delay times $\tilde{\tau}_{jk} = \tau_{jk} + \eta_k - \eta_j$. To avoid the creation of negative delays, we demand $\eta_j - \eta_k \leq \tau_{jk}$. Under this constraint, we can prove an equivalence between solutions of the original system (1) and the transformed system (4). In particular, there is a natural one-to-one correspondence between the ω-limit-sets of both systems such that corresponding sets possess the same type of stability [18].

As a next step we compute possible delay reductions in simple motifs which might be thought of as building blocks for more complex networks. This will provide a picture of the possibilities and restrictions of the reduction method. Each example consists of one semicycle. This is an undirected cycle of the underlying network graph. The graphs of the considered motifs are shown in Fig. 2.

4. Motif A

The dynamics on network A are described by

$$\begin{aligned} \dot{x}_1(t) &= f_1(x_1(t)), \\ \dot{x}_2(t) &= f_2(x_2(t), x_1(t - \tau_1), x_1(t - \tau_2)). \end{aligned}$$

and the transformed system which corresponds to the timeshifts η_1 and η_2 possesses the new delays $\tilde{\tau}_j = \tau_j + \eta_1 - \eta_2$, $j = 1, 2$. The choice $\eta_2 - \eta_1 = \tau_1$ yields $\tilde{\tau}_1 = 0$ and $\tilde{\tau}_2 = \tau_2 - \tau_1$ which may be assumed non-negative without loss of generality. This transformed system has only one connection delay – the second connection being instantaneous now. If the delays are the same, the situation becomes formally one with only one connection between both nodes whose delay $\tau = \tau_1 = \tau_2$ can be neglected by the choice $\eta_2 - \eta_1 = \tau$.

5. Motif B

The transformed delays in case of motif B are $\tilde{\tau}_1 = \tau_1 + \eta_1 - \eta_2$, $\tilde{\tau}_2 = \tau_2 + \eta_1 - \eta_3$ and $\tilde{\tau}_3 = \tau_3 + \eta_2 - \eta_3$. Again, there exists at least one choice of η_j which transforms the equations to contain only one delay time. If $\tau = \tau_1 - \tau_2 + \tau_3 \geq 0$, we can require that all new delays are identical $\tilde{\tau}_1 = \tilde{\tau}_2 = \tilde{\tau}_3 \equiv \tau$. On the other hand, if $\tau < 0$ then the previous requirement cannot be satisfied. In this case the only way to achieve a reduction to one delay is to choose $\tilde{\tau}_2 = -\tau$. That implies $\tilde{\tau}_2 = \tilde{\tau}_1 = 0$. Simple calculations yield five different reductions to a single delay of which four apply to the case $\tau \geq 0$ and one to the case $\tau \leq 0$. In the case $\tau = 0$ all delays can be eliminated and the equations are equivalent to an ordinary differential equation.

6. Motif C

Analogously, one can transform the delays of motif C, τ_1, τ_2, τ_3, and τ_4, such that only a single delay τ is needed to describe the dynamics. There are ten different choices of timeshifts which lead to such a reduction. The resulting delays $\tilde{\tau}_j$ are listed in Table 1 in [17]. Again, we find a

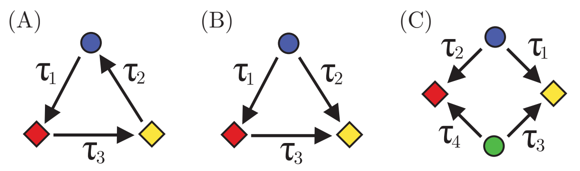

Figure 2: Exemplary motifs with connection delays. Each has a cycle space dimension equal to one.

characteristic delay sum $\tau = \tau_1 - \tau_2 - \tau_3 + \tau_4$ which is most suitable to describe the possible reductions. There are five possible reductions for the case $\tau \geq 0$ and five for the case $\tau \leq 0$. In contrast to motif B, if $\tau \neq 0$ one cannot reduce the delays in a way that all links hold the same delay.

7. The generalized roundtrip

For each of the motifs A–C, we call the value $\mathbf{rt} = |\tau|$ the (*generalized*) *roundtrip*. Like the roundtrip which was mentioned for the unidirectional ring (2), it is a sum of the delays along a given semicycle. The generalized roundtrip additionally takes into account the direction of the links $k \to j$ which determines whether the corresponding delays enter the sum as $+\tau_{jk}$ or $-\tau_{jk}$. For a given semicycle $c = (\ell_1, ..., \ell_k)$ which is a closed, undirected path along the links $\ell_1, ..., \ell_k$, we fix an orientation and define the generalized roundtrip of c as

$$\mathbf{rt}(c) = \left| \sum_{j=1}^{k} \sigma_j \tau(\ell_j) \right|, \tag{5}$$

where $\tau(\ell)$ is the delay of the link ℓ_j and $\sigma_j = \pm 1$ depending on whether the direction of the j-th link coincides with the orientation or not. See Fig. 3 for an illustration. Since the modulus of the sum is taken, it is independent of the chosen reference orientation. It can easily be checked that the roundtrip $\mathbf{rt}(c)$ is preserved under the timeshift (3) for any semicycle c.

8. Delay reduction for general coupling topologies

The motifs considered above point to the general result that the delays on an arbitrary semicycle c can be reduced to a single delay. It is not always possible to do so by homogeneously distributing the roundtrip of the cycle along its links as we have seen for motif A and C. But it is indeed possible to concentrate the roundtrip delay time on a single link. Anyhow, when proceeding from motifs with cycle dimension one towards more complex coupling topologies, one has to take into account the interplay between several, possibly interconnected semicycles. Without going into details of the proof, we state a result which classifies networks of the form (1) and provides a normal form which involves a minimal number of distinct delays. We are able to show that for any system of the form (1), there exists a spanning

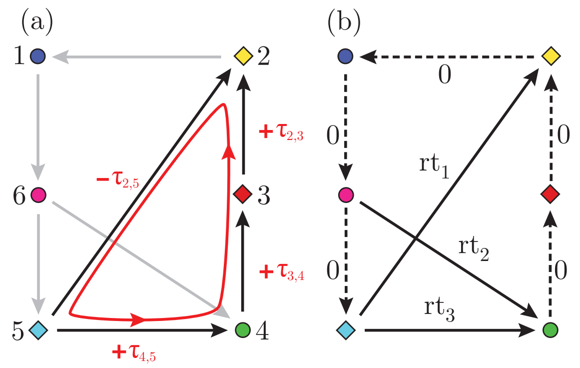

Figure 3: (a) The generalized roundtrip of a semicycle c (solid links) is the modulus of the oriented sum of the delays along the cycle's links [Eqn. (5)]. An orientation is indicated by the red curve and the corresponding weights σ_j by the symbols '±'. (b) A possible choice for a spanning tree S is the set of dashed links. Any system of the form (1) with the depicted topology is equivalent to a system with instantaneous connections in S. The delay-times $\mathbf{rt}_j$ on the fundamental links are equal to the roundtrips along the corresponding fundamental cycles.

tree $S = \{\ell_1, ..., \ell_{N-1}\}$ (i.e., a semicycle-free set of $N-1$ links) and timeshifts η_j, $1 \leq j \leq N$, such that

(i) connections $\ell \in S$ are instantaneous in the transformed system, i.e.,

$$\tau(\ell) = 0, \text{ for } \ell \in S,$$

(ii) connections $\ell \notin S$ hold a non-negative delay

$$\tau(\ell) = \mathbf{rt}\,(c(\ell)), \text{ for } \ell \notin S,$$

where $c(\ell)$ is the semicycle which is created by adding $\ell \notin S$ to S [Fig. 3].

Consequently, the essential number of delays is equal to the cycle space dimension $C = L - N + 1$ of the network.

Acknowledgments

We thank K. Knauer and M. Zaks for useful discussions and we acknowledge the financial support by the DFG in the framework of the Collaborative Research Center SFB910.

References

[1] L. Appeltant, M. C. Soriano, G. Van der Sande, J. Danckaert, S. Massar, J. Dambre, B. Schrauwen, C. R. Mirasso, and I. Fischer. Information processing using a single dynamical node as complex system. *Nature Comm*, 2, 2011.

[2] A. Argyris, D. Syvridis, L. Larger, V. Annovazzi-Lodi, P. Colet, I. Fischer, J. Garcia-Ojalvo, C. R. Mirasso, L. Pesquera, and K. A. Shore. Chaos-based

communications at high bit rates using commercial fibre-optic links. *Nature*, 438(7066):343–346, Nov. 2005.

[3] P. Baldi and A. Atia. How delays affect neural dynamics and learning. *IEEE Transactions on Neural Networks*, 5:1045–9227, 1994.

[4] C. E. Carr. Processing of temporal information in the brain. *Annu. Rev. Neurosci.*, 16:223–243, 1993.

[5] G. V. der Sande, M. C. Soriano, I. Fischer, and C. R. Mirasso. Dynamics, correlation scaling, and synchronization behavior in rings of delay-coupled oscillators. *Physical Review E (Statistical, Nonlinear, and Soft Matter Physics)*, 77(5):055202, 2008.

[6] O. D'Huys, R. Vicente, T. Erneux, J. Danckaert, and I. Fischer. Synchronization properties of network motifs: Influence of coupling delay and symmetry. *Chaos*, 18(3):037116, 2008.

[7] T. Erneux. *Applied Delay Differential Equations*, volume 3 of *Surveys and Tutorials in the Applied Mathematical Sciences*. Springer, 2009.

[8] V. Flunkert, S. Yanchuk, T. Dahms, and E. Schöll. Synchronizing distant nodes: A universal classification of networks. *Phys. Rev. Lett.*, 105(25):254101, Dec 2010.

[9] J. Foss, A. Longtin, B. Mensour, and J. Milton. Multistability and delayed recurrent loops. *Phys. Rev. Lett.*, 76:708–711, 1996.

[10] A. L. Franz, R. Roy, L. B. Shaw, and I. B. Schwartz. Effect of multiple time delays on intensity fluctuation dynamics in fiber ring lasers. *Phys. Rev. E*, 78(1):016208, 2008.

[11] G. Giacomelli and A. Politi. Relationship between delayed and spatially extended dynamical systems. *Phys. Rev. Lett.*, 76(15):2686–2689, 1996.

[12] J. K. Hale and S. M. Tanaka. Square and pulse waves with two delays. *Journal of Dynamics of Differential Equations*, 12:1–30, 2000.

[13] S. Heiligenthal, T. Dahms, S. Yanchuk, T. Jüngling, V. Flunkert, I. Kanter, E. Schöll, and W. Kinzel. Strong and weak chaos in nonlinear networks with time-delayed couplings. *Phys. Rev. Lett.*, 107:234102, 2011.

[14] I. Kanter, M. Zigzag, A. Englert, F. Geissler, and W. Kinzel. Synchronization of unidirectional time delay chaotic networks and the greatest common divisor. *EPL (Europhysics Letters)*, 93(6):60003, 2011.

[15] C. Leibold and J. L. van Hemmen. Spiking neurons learning phase delays: How mammals may develop auditory time-difference sensitivity. *Phys. Rev. Lett.*, 94(16):168102, Apr. 2005.

[16] X. Li, A. B. Cohen, T. E. Murphy, and R. Roy. Scalable parallel physical random number generator based on a superluminescent led. *Optics Lett.*, 36:1020, 2011.

[17] L. Lücken, J. Pade, and S. Yanchuk. Reduction of interaction delays in networks. arXiv:1206.1170, 2012.

[18] L. Lücken, J. P. Pade, S. Yanchuk, and K. Knauer. in preparation.

[19] R. M. Memmesheimer and M. Timme. Designing complex networks. *Physica D*, 224(1-2):182–201, Dec. 2006.

[20] P. Perlikowski, S. Yanchuk, O. V. Popovych, and P. A. Tass. Periodic patterns in a ring of delay-coupled oscillators. *Phys. Rev. E*, 82(3):036208, Sep 2010.

[21] O. V. Popovych, S. Yanchuk, and P. A. Tass. Delay- and coupling-induced firing patterns in oscillatory neural loops. *Phys. Rev. Lett.*, 107:228102, 2011.

[22] R. Vicente, S. Tang, J. Mulet, C. R. Mirasso, and J.-M. Liu. Synchronization properties of two self-oscillating semiconductor lasers subject to delayed optoelectronic mutual coupling. *Phys. Rev. E*, 73:047201, 2006.

[23] M. Wolfrum, S. Yanchuk, P. Hövel, and E. Schöll. Complex dynamics in delay-differential equations with large delay. *Eur. Phys. J. Special Topics*, 191:91–103, 2010.

[24] S. Yanchuk. Discretization of frequencies in delay coupled oscillators. *Phys. Rev. E*, 72:036205, 2005.

[25] S. Yanchuk and P. Perlikowski. Delay and periodicity. *Phys. Rev. E*, 79(4):046221, 2009.

[26] S. Yanchuk, P. Perlikowski, O. V. Popovych, and P. A. Tass. Variability of spatio-temporal patterns in non-homogeneous rings of spiking neurons. *Chaos*, 21:047511, 2011.