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Adaptive Delayed Feedback Control Algorithm With an Iterated Feedback Gain in the Presence of Noise

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Abstract—We propose an adaptive modification of the delayed feedback control (DFC) method that automatically finds the optimal feedback gain of the DFC systems. The system under control is perturbed by a Gaussian white noise with a low amplitude, and the variance of the delayed difference is estimated by a simple integrator. The problem of minimization of the variance is substituted by a minimization of some quantity that represents a generalization of the Lyapunov exponent in the presence of noise. The generalization is derived by considering a simple Langevin equation. The numerical simulations for the controlled Rössler system show that the adaptively obtained optimal feedback gains are in good quantitative agreement with the corresponding exact values.

1. Introduction

Although delayed feedback control algorithm has been introduced two decades ago [1] it is still one of the most active fields in applied nonlinear science [2]. This algorithm provides a simple, robust, and efficient tool for stabilization of unstable periodic orbits (UPOs) in nonlinear dynamical systems. The control signal in the DFC algorithm is formed from a difference between the current state of the system and the state of the system delayed by one period of a target orbit. Such control signal allows one to treat the controlled system as a black box; it does not require any exact knowledge of either the form of the periodic orbit or the system's equations. The method is asymptotically noninvasive because the control force vanishes whenever the target UPO is reached. The DFC algorithm has been successfully implemented in quite diverse experimental systems from different fields of science. Some details of experimental implementations as well as various modifications of the DFC algorithm can be found in the review paper [3].

One of the relevant issues in the application of the DFC method is the search for the delay time, which should be equal to the period of actual UPO. This problem was solved by using e.g. the gradient descent method (see [4] and references therein). Here we consider an another problem: the adaptive search for the optimal value of the feedback gain in the presence of noise. For the noiseless systems the optimal feedback gain corresponds to the minimum of the leading Lyapunov exponent (LE). However, in the presence of noise, the LEs are not available. In this case one can invoke the minimization of the variance of the delayed difference. It appears that the variance of the delayed difference is an unappropriate parameter since it has a flat minimum and singularities at the boundaries of stability. Instead of the variance we introduce a new quantity that preserves the same minimum but has more appropriate properties for the construction of the adaptive algorithm.

The noise plays a crucial role in our algorithm. Without noise, the variance of delayed difference is exactly zero in the stable range of feedback gain. There is a flat minimum present, and one has nothing to optimize. In contrast, in the presence of noise, the final variance of delayed difference (after successful stabilization) is non-zero, and it is less for better feedback gains, and larger for the worse values of them. There exists an optimal value of feedback gain that minimizes the amplitude of control signal.

It is noteworthy that in [5, 6, 7] there has been substantially studied the role of noise in the coupled logistic maps controlled by generalized DFC schemes. In these analysis there was suggested a criterion for estimating the noise level that can be tolerated by the given controller. It was also shown that the DFC controller equivalent to standard optimal controller can be equally robust in the presence of noise, and that the NDFC controller tends to be less sensitive to noise than the extended DFC (EDFC).

In [8] there has been proposed an adaptive algorithm for tuning of feedback gain in DFC systems. This algorithm is based on the speed-gradient method that enables to minimize the goal function defined as a squared delayed difference. Beginning from a zero initial value the variable feedback gain converges towards an appropriate value lying inside of the stability range. The final value of the feedback gain depends on the initial conditions of the system and on the adaptation gain. The modification is able to stabilize the target even if the stability interval is unknown.

In [9] it was also demonstrated the efficiency of the speed-gradient method for adaptive synchronization in delayed-coupled networks of Stuart-Landau oscillators. By proper choice of the coupling phase one can switch between different synchronous oscillatory states of the network. The authors have proposed goal functions based e.g. on a generalized order parameter and demonstrated that the speed-gradient method allows one to find appropriate coupling phases with which different states of synchronization can be selected.

We should also mention that in [10] an adaptive algorithm was proposed for locating unknown steady states. This technique includes both dynamical estimators and coupling gains. The authors wrote down the dynamic equation for the feedback gain whose r.h.s. consists of the squares of the estimated difference for each system variable. In this method the growth of feedback gain is unbounded. Therefore the result does not converge to the optimal value of feedback gain.

An analogous adaptive rule was applied in [11] where the problem of stabilization of the UPOs by DFC method was solved. In this case the both DFC parameters (delay and feedback gain) were found adaptively by additional dynamic equations. The r.h.s. of the equation for the feedback gain consists of the squared delayed difference. This rule causes the unbounded growth of the feedback gain, and, as a result, the period of the target orbit is found only approximately.

2. The Idea of Algorithm

In order to illustrate our idea, first consider a simple Langevin equation

$$\dot{x} = -\gamma x + \xi(t), \tag{1}$$

that describes the dynamics of a stable fixed point subjected to noise. Here $-\gamma$ is the eigenvalue of the fixed point (the Lyapunov exponent), and $\xi(t)$ is the white Gaussian noise satisfying

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t)\xi(t') \rangle = \varepsilon^2 \delta(t - t'). \tag{2}$$

where $\langle\rangle$ denotes the averaging over ensemble, and ε is the strength of the noise. Our aim is to find the asymptotic value of the variance $\langle x^2 \rangle$. Solving the Eq.(1) by variation of constant one gets

$$\langle x^2(t) \rangle = \langle x_0^2 \rangle e^{-2\gamma t} + \frac{\varepsilon^2}{2\gamma} - \frac{\varepsilon^2}{2\gamma} e^{-2\gamma t}. \tag{3}$$

Here $\langle x_0^2 \rangle$ denotes the averaging over initial values of $x^2(0)$. For $t \to \infty$ the asymptotics reads:

$$\langle x^2(t) \rangle_{t\to\infty} = \varepsilon^2/2\gamma. \tag{4}$$

Thus the eigenvalue of the fixed point can be determined from the variance as follows

$$-\gamma = -\varepsilon^2/2\langle x^2(t) \rangle_{t\to\infty}. \tag{5}$$

Below we utilize this expression to introduce a convenient quantity for the optimization of the DFC algorithm.

Now consider the DFC controlled dynamical system in the presence of noise:

$$\dot{\boldsymbol{X}}(t) = \boldsymbol{F}[\boldsymbol{X}(t), k\Delta s(t)] + \boldsymbol{\xi}(t). \tag{6}$$

The first argument in the function $\boldsymbol{F}$ shows the dependence of the vector field on internal degrees of freedom, while the second argument denotes the dependence on control force $k\Delta s(t)$. Here k is the feedback gain, and the delayed difference is defined as $\Delta s(t) = s(t) - s(t - \tau) = g[\boldsymbol{X}(t)] - g[\boldsymbol{X}(t - \tau)]$. Here $s(t) = g[\boldsymbol{X}(t)]$ is a measurable scalar signal that is a function of all the variables. $\boldsymbol{\xi}(t) \equiv \{\xi_\alpha(t)\}$ is a vector of Gaussian white noise satisfying $\langle \xi_i(t) \rangle = 0$ and $\langle \xi_i(t)\xi_j(t') \rangle = \varepsilon^2 \delta_{ij} \delta(t - t')$. The free system ($k = 0$, $\varepsilon = 0$) has an unstable periodic solution $\boldsymbol{X}(t) = \boldsymbol{\eta}(t) = \boldsymbol{\eta}(t - T)$ that we intend to stabilize by control perturbation $k\Delta s(t)$. Our aim is to find the optimal value of the feedback gain k_{op} that corresponds to the minimal asymptotic variance of the delayed difference.

Denoting the variance of delayed difference as $D^2 = \langle [s(t) - s(t - \tau)]^2 \rangle$ we introduce a quantity

$$L = -\varepsilon^2/2D^2, \tag{7}$$

which is analogous to the above definition of the Lyapunov exponent of the fixed point [cf. Eq.(5)]. Although we now deal with the periodic orbit rather than the fixed point we can still imagine that L is a good characteristic of the Lyapunov exponent of the stabilized UPO. Subsequently, we reformulate our aim as follows. Now we will seek such a value of the feedback gain k_{op} that corresponds to the minimum of quantity L. We have made such a change because the shape of the function $L = L(k)$ is much more convenient for optimization than for $D^2 = D^2(k)$, whereas the minimum remains at the same value k_{op}.

Now we outline our algorithm for iterative search of the optimal feedback gain. Suppose, we know the boundaries of stability of the controlled orbit, and denote them as $k \in (k_{min}, k_{max})$. Let us assume that we start from the value of $k = k_0$ that lies near the k_{max}. We choose an appropriate positive constant β, and construct a sequence of feedback gains:

$$k_N = k_0, \quad k_{N-1} = k_0 - \beta, ..., k_1 = k_0 - (N - 1)\beta. \tag{8}$$

We integrate numerically the system (6) together with the following equation:

$$\dot{w} = \alpha_{sw}[s(t) - s(t - \tau)]^2. \tag{9}$$

Here the parameter α_{sw} is used to turn on the procedure of integration (we set $\alpha = 0$ for $t \leq t_{end} - T_\alpha$ and $\alpha = 1$ otherwise).

We set $w(0) = 0$, and thus the quantity $w(t_{end})/T_\alpha$ will yield the averaged variance of the delayed difference, i.e.

$$w(t_{end})/T_\alpha \simeq \langle [s(t) - s(t - \tau)]^2 \rangle. \tag{10}$$

Note that t_{end} (the end of numerical integration), and T_α (determining the moment at which we begin to integrate the equation (9)) must be large enough in order to get adequate values of $w(t_{end})$ which would satisfy (10).

We thus may redefine the quantity L as

$$L(k) = -\frac{\varepsilon^2 T_\alpha}{2w(t_{end})}. \quad (11)$$

This definition is obtained by substituting $D^2 \rightarrow w(t_{end})/T_\alpha$ in (7).

We then integrate the system (6) until t_{end} for each value of $\{k_1, ..., k_N\}$ from (8) and compute the corresponding values of $w(t_{end})$ thus obtaining the sequence

$$w^N, \quad w^{N-1}, ..., w^1. \quad (12)$$

Here the upper indices of w^n numerate the final values of w for corresponding values of k_n, and we have omitted the argument (t_{end}). Afterwards, using (11) we compute the sequence

$$L_N, \quad L_{N-1}, ..., L_1. \quad (13)$$

We now fit the obtained points $\{k_n, L_n\}$ by using the least squares method (LSM). This method yields the coefficients (S_0, S_1, S_2) of parabola $L = S_0 + S_1 k + S_2 k^2$ that approximates the obtained sequence. The minima of parabola is given simply by

$$k_{op} = -S_1/2S_2. \quad (14)$$

After that we construct the new sequence of feedback gains (note that N must be an odd number):

$$k_{op} - \beta(N-1)/2, \quad k_{op} - \beta(N-1)/2 + \beta, ..., k_{op} + \beta(N-1)/2. \quad (15)$$

For each feedback gain of this sequence we compute the values of L_n thus obtaining again (13).

Given the sequence $\{k_n, L_n\}$, we compute the parameters (S_0, S_1, S_2) again, and find the new optimal k_{op} as given by (14).

By repeating the above procedure, we get the dynamics of k_{op} that should converge towards the exact optimal value.

Note that we can use quadratic approximation of $L = L(k)$ for simple systems, whereas for more complicated cases we use the cubic parabola.

3. Controlling the Rössler System

As an illustrative example consider the DFC controlled Rössler system under the noise (its free version was introduced in [12]):

$$\dot{x} = -y - z + \xi_x(t), \quad (16)$$
$$\dot{y} = x + ay - k[y(t) - y(t-\tau)] + \xi_y(t), \quad (17)$$
$$\dot{z} = b + z(x - c) + \xi_z(t). \quad (18)$$

Here $a = b = 0.2$ and $c = 5.7$ are the system parameters, $k[y(t) - y(t-\tau)]$ is the feedback perturbation, and k is the feedback gain. The vector $\boldsymbol{\xi}(t) = [\xi_x(t), \xi_y(t), \xi_z(t)]$ represents the noise perturbations that satisfy $\langle \xi_\alpha(t) \rangle = 0$, $\langle \xi_\alpha(t)\xi_{\alpha'}(t') \rangle = \varepsilon^2 \delta(t - t')$ in which $\alpha, \alpha' = x, y, z$.

The system parameters are chosen such as to get a chaotic behavior in the absence of noise and feedback perturbation (for $\varepsilon = 0$ and $k = 0$). Our aim is to stabilize the period-1 UPO (with period $T_1 = 5.88105$) of the Rössler system and to find the optimal feedback gain k_{op}.

3.1. Adaptive Search for the Optimal Feedback Gain in the Controlled Rössler System

In Sec.2 we have described the iterative procedure for finding the optimal feedback gain in the general case. Here we use the same procedure to a specific case, namely, to the Rössler system. In Sec.2 we have adopted the fitting of a quadratic parabola for the sequence $\{k_n, L_n\}$ whereas for the Rössler system we use the fitting of the cubic parabola.

Let us discuss the procedure for finding the quantity L that depends on k. We integrate numerically the system (16,17,18) together with equation (9) with $s(t) = y(t)$.

We set $w(0) = 0$, and thus the quantity $w(t_{end})/T_\alpha$ will again yield the averaged variance of the delayed difference, i.e.

$$w(t_{end})/T_\alpha \simeq \langle [y(t) - y(t-\tau)]^2 \rangle. \quad (19)$$

The quantity L is defined by equation (11).

Hence, given the sequence $\{k_n, L_n\}$, we approximate these points by a cubic parabola $L = S_0 + S_1 k + S_2 k^2 + S_3 k^3$. The LSM yields the coefficients (S_0, S_1, S_2, S_3). We then need one of two extremes of this parabola, namely, the minimum. The first derivative of the cubic parabola is zero at

$$k_\pm = [-S_2 \pm (S_2^2 - 3S_3 S_1)^{1/2}]/3S_3. \quad (20)$$

In our case, the necessary minimum corresponds to the k_+, and we thus get the optimal feedback gain as

$$k_{op} = k_+ = [-S_2 + (S_2^2 - 3S_3 S_1)^{1/2}]/3S_3, \quad (21)$$

whereas k_- corresponds to the left maximum which is outside of the stability domain.

By using this algorithm we obtain the iterated dynamics of optimal feedback gain k_{op} which should converge towards the exact minimum of $L = L(k)$.

If we dealt with some linear maps, we would obtain the dependencies $L = L(k)$ analytically. However, for the controlled Rössler system, we have encountered a more complicated situation. Since the analytic dependence of $L = L(k)$ is not available now, we do not have any analytic exact value of the optimal feedback gain. However, the iterations of optimal feedback gain have revealed that the variance of the obtained values in respect to their average is relatively small, i.e. the obtained points are located in a close neighborhood to the average.

4. Conclusions

We have proposed an adaptive modification of the delayed feedback control algorithm that enables the controller to automatically find the optimal feedback gain of the DFC systems.

This modification uses the quantity L that depends on the feedback gain. The minimum of this quantity corresponds to the optimal case when the variance of the delayed difference is minimal. The expression of the quantity L is based on the considerations of the simplest case, the Langevin equation (1), where it has the meaning of the eigenvalue of the steady state.

One may also remember the recent adaptive algorithm [4] for finding the delay time which should be equal to the exact period of the target orbit. In that case we did not need to involve any noise since the incorrect delays (non-equal to the exact periods) produced the variance of the delayed difference with resonant minima at the exact periods. Therefore we conclude that the problem of search for optimal feedback gain is much more complicated than that of the delay.

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