IEICE Proceeding Series

Chaotic delayed maps and their natural measure

Valentin Flunkert, Ingo Fischer

Vol. 1 pp. 755-758
Publication Date: 2014/03/17
Online ISSN: 2188-5079

2012 International Symposium on Nonlinear Theory and its Applications
NOLTA2012, Palma, Majorca, Spain, October 22-26, 2012

Chaotic delayed maps and their natural measure

Valentin Flunkert[†] and Ingo Fischer[†]

†Instituto de Fisica Interdisciplinar y Sistemas Complejos,
IFISC (UIB-CSIC), Campus Universitat de les Illes Balears, E-07122 Palma de Mallorca, Spain
Email: flunkert@itp.tu-berlin.de, ingo@ifisc.uib-csic.es

Abstract—The natural measure of a chaotic attractor describes many statistical properties of the system. It is well known that the natural measure is related to the unstable periodic orbits in the attractor. Here, we investigate the natural measure of simple delayed maps. We argue that for large delay there are two types of unstable periodic orbits that need to be considered: (i) periodic orbits with periods much smaller than the delay and (ii) periodic orbits with periods close to multiples of the delay time. In a space time representation of the dynamics the latter orbits correspond to unstable pulse-like solutions. Our results suggest that in the limit of large delay times, the natural measure converges in the sense of the spatio-temporal interpretation of the delay system.

1. Introduction

Dynamical systems with delays occure in a many different scientific fields, such as biology, physics, engineering, or socio-economics [1]. In a delayed system the evolution of the system does not only depend on the present state of the system, but also on the history. In the simplest case the state variables $X(t)$ evolve as

$$\dot{X}(t) = F[X(t), X(t-\tau)] , \tag{1}$$

where τ is a discrete delay.

Delayed systems have a vast range of applications in particular in engineering. Many systems with self-feedback can modeled by equations of the form of Eq. (1). Examples of such systems are optical and optoelectronic systems [2, **?**, 3, 4, 5], or biological processes [6].

Instead of a delay differential equation with a continuous time one can also study the discrete time case of a delayed map

$$x(t) = f\,[x(t-1),\, x(t-\tau)] \qquad (x \in \mathbb{R}^n) , \tag{2}$$

where $t \in \mathbb{Z}$ is the discrete time and $\tau \in \mathbb{N}$ is the value of the delay. Delayed maps and delay differential equations share have many similarities [7]. In particular, Eq. (1) can always be approximated by Eq. (2) as is usually done in the numerical simulation.

Systems with delayed self-feedback, such as Eqs. (1) and (2) usually become chaotic for large values of the delay time. In chaotic regime, many characteristic properties of chaotic attractor are then independent of the value of the delay [8, 9] τ, when τ is large enough. For instance, the Kolmogorov- Sinai entropy converges to a constant for $\tau \to \infty$. On the other hand other properties scale proportionally to τ. The dimensionality of the attractor, for instance, grows linearly with τ.

In this work we aim to understand generic aspects of Eq. (2) by studying unstable periodic solutions in these systems.

In particular, we show that there are two important classes of periodic orbits in a delayed system with large delay: (i) periodic orbits with small period that repeat as the delay is increased and (ii) periodic orbits with a period close to the delay time. The later type of orbits are pulse like solutions. The pulse like orbits are in a chaotic system unstable, but may be stable in other dynamical

Further, we discussing the role of these orbits for the natural measure of the chaotic system. This discussion is so only intuitive and not rigorous. Nevertheless, we believe that it could be a valuable approach for further investigations.

2. Analysis

We consider the delayed map of Eq. (2) for simplicity in one dimension, i.e., $x(t) \in \mathbb{R}$:

$$x(t) = f\,[x(t-1),\, x(t-\tau)] , \tag{3}$$

where $t \in \mathbb{Z}$ is the discrete time and $\tau \in \mathbb{N}$ is the value of the delay.

Natural measure — Consider a chaotic d-dimensional map

$$Y(t+1) = M\,[Y(t)] \tag{4}$$

$Y(t) \in \mathbb{R}^d$. Grebogi, Ott, and Yorke showed in their seminal paper [10] that if the map M is hyperbolic, the natural measure of a set S can be expressed as

$$\mu(S) = \lim_{p\to\infty} \sum_{Y_p(j)\in S} \frac{1}{L_u\big(Y_p(j)\big)} . \tag{5}$$

Here, $Y_p(j)$ is the j-th fixed point of the p times iterated map M^p, i.e., the sum goes over all periodic orbits with period p or factors of p. The term L_u is given by

$$L_u = |\lambda_1 \cdot \lambda_2 \cdots \lambda_n| , \tag{6}$$

where λ_i are the unstable eigenvalues ($|\lambda_i| > 1$) of the Jacobian $DM^p(Y_p(j))$. Although Eq. (5) has only been proved for hyperbolic maps, strong evidence [11, 12] have been presented that support the validity of Eq. (5) for many systems also in the non-hyperbolic case. In the following, we assume that Eq. (5) holds for our delayed map (3).

We can consider the delayed system Eq. (3) as a map M acting on the τ dimensional state space I^τ (with $x_k \in I$) corresponding to the state vectors $Y(t) = (Y_1(t), \ldots, Y_\tau(t)) := (x(t-1), \ldots, x(t-\tau))$. It is straight forward to find the action of (3) on $Y(t)$.

Consider now a T-periodic orbit of Eq. (3). A Floquet multiplier z of this orbit then corresponds to the eigenvalue of the Jacobian of the T-times iterated map M^T evaluated on any point of the orbit. Similarly, for $p = rT$ ($r \in 1, 2, \ldots$), the eigenvalues of DM^p correspond to z^r.

Periodic orbits with small periods — For a T-periodic orbit of the delayed map (3), the variational equation is given by

$$\xi(t) = A(t)\xi(t-1) + B(t)\xi(t-\tau) , \tag{7}$$

where $A(t)$ and $B(t)$ are T-periodic Jacobian matrices. Making a Floquet-like ansatz $\xi(t) = z^t q(t)$, where $q(t)$ is T-periodic, yields

$$zq(t) = A(t)q(t-1) + z^{-\tau}B(t)q(t-\tau) . \tag{8}$$

We consider the case of large delay and small period ($T \ll \tau$). In this case it follows [13], that the solutions z are given by isolated strongly unstable solutions $\hat{z}_k$ independent of τ and by branches $z_i = \left(1 + \delta(\omega_i)/\tau\right)e^{i\omega_i}$, where $\delta(\omega)$ is the curve on which the solutions lie. The allowed values ω_i are spaced as

$$\omega_{i+1} - \omega_i = 2\pi/\tau + O(1/\tau^2) . \tag{9}$$

The weight of an orbit — We now calculate the factor $L_u(Y_p(j))$ for a periodic orbit with period $T = p \ll \tau$. Consider first the case, when all eigenvalues lie on a single branch $\delta(\omega)$. Taking the logarithm of $L_u(Y_p(j))$ gives $\ln L_u(Y_p(j)) = \sum_{|z_i|>1} \ln |z_i|$, where the sum goes over all unstable eigenvalues. Using the frequency spacing Eq. (9), we can express the sum as an integral over frequencies $\omega \in W_+$, where $W_+ = \{\omega \mid \delta(\omega) > 0\}$. Thus we find for large delays

$$\begin{aligned}\ln L_u(Y_p(j)) &\approx \sum_{|z_i|>1} \frac{\Delta\omega}{\Delta\omega} \ln |1 + \delta(\omega_i)/\tau| \\ &\approx \frac{1}{2\pi} \int_{W_+} d\omega\, \delta(\omega) ,\end{aligned}$$

which gives

$$L_u(Y_p(j)) \approx \exp\left[\frac{1}{2\pi} \int_{W_+} d\omega\, \delta(\omega)\right] . \tag{10}$$

The case of multiple branches (for $\delta_k(\omega)$ and strongly unstable solutions Λ_i can be handled similarly.

The main result from this analysis is that for a given order p in Eq. (5), in the limit of large delay times the weight of each periodic orbit is independent of τ.

3. Pulse like periodic orbits

Apart from the periodic orbits with periods small compared with the delay time, there typically exist periodic orbits whose periods are close to multiples of the delay time ($T = k \cdot \tau + \Delta$, with small $|\Delta|$). As we will discuss below, these orbits occur for large delay and are pulse-like solutions. Consider again the delayed map Eq. (3). We cast the delayed map into a space-time representation

$$x_k(t) = f\left[x_k(t-1),\ x_{k-1}(t)\right] , \tag{11}$$

where $k \in \mathbb{N}$ is the time-like coordinate and $t \in \{1, \ldots, \tau\}$ is the space-like coordinate. The feature of the delay come in through the particular boundaries conditions $x_k(t-\tau) = x_{k-1}(t)$. For large delay one may expect that the particular boundary conditions can be neglected at least for certain solutions.

Let us thus consider the dynamics induced by Eq. (11) on the whole axis $t \in \mathbb{Z}$. We look for travelling pulse like solutions that have an invariant shape. These solutions thus obey

$$x_k(t) = Y(t - k\Delta) , \tag{12}$$

where Δ corresponds to the speed of the pulse. General solutions of this form will not necessarily be solutions of the delayed equation due to the different boundary condition. However, localized solutions, will be approximate solutions of the delay equation for large delay, since the boundary conditions are almost satisfied due to the localization.

Substituting Eq. (12) in (11) gives an advanced-delay equation

$$Y(t) = f\left[Y(t-1),\ Y(t+\Delta)\right] . \tag{13}$$

A homoclinic solution to a fixed point x_* of this equation would give the pulse solution of the original system in the limit $\tau \to \infty$. As we numerically demonstrate below, a homoclinic solution is accompanied by many periodic orbits close to the homoclinic solution. Such a periodic solution with a period T, corresponds to a pulse like solutions of our original system with delay $\tau = T + \Delta$. Of course not for all Δ a solution must exist. Note that the occurrence of these orbits close to the homoclinic solution is the same phenomenon as in partial differential equations, for which periodic travelling waves with large spatial period appear in a similar manner [14].

The pulse solutions have a period $T = \tau + \Delta$ close to the value of the delay. Similarly, there may exist solutions with a period $T = n\tau + \Delta$ close to multiples $n\tau$ of the delay ($n \in \mathbb{N}$). These solutions correspond to localized profiles that repeat after n iterations shifted by Δ, i.e., $x_{k+n}(t) = x_k(t - \Delta)$. In the intermediate iterations the profile does not match previous profiles.

For our numerical investigations, we consider a delayed map of the following type

$$x(t) = \epsilon f\left(x(t-1)\right) + (1-\epsilon) f\left(x(t-\tau)\right) , \tag{14}$$

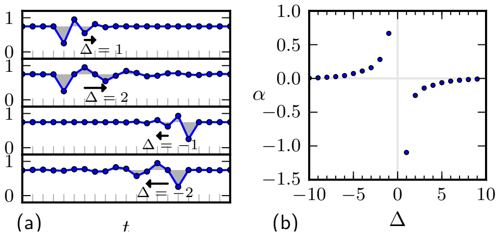

Figure 1: Panel (a): Exemplary unstable pulse solutions for different values of Δ. Panel (b): Decay rate α of front of pulse solutions vs. speed Δ. Parameters: $\epsilon = 0.12$, $r = 3.95$.

with $\epsilon \in [0, 1]$. As the local map we have investigated the logistic map, the Bernoulli map and the tent map. The results are similar and we thus limit the discussion to the tent map $f(x) = r \min\{x, 1 - x\}$. To find the pulse solutions with speed Δ and iteration number $n = 1$ for a system with delay τ numerically, we solve Eq. (13) on a lattice of size $T = \tau + \Delta$ with periodic boundary conditions using the Newton-Raphson method. As an initial condition we take the (non-zero) fixed point of the logistic map $x(t) = x_*$ and perturb this homogeneous steady state at one or more lattice sites. In our numerical experiments this initial condition converges with a high probability to a one- or multi-pulse solution. We find pulse solutions for $n > 1$ numerically in a similar way by considering a lattice of size $T = n\tau + \Delta$ with periodic boundary conditions.

Fig. 1a shows the pulse shapes for different values of Δ and n. All pulses we found have a front in the direction of propagation that decays exponentially towards the fixed point value x_* and a back with a sharp transition to the steady state value x_*. The exponential decay rate α of the front of the pulse depends on the speed of the pulse. Pulses with large speeds (large values of $|\Delta|$) have a slower decay and are thus wider. This is depicted in Fig. 1b, where the decay rate α of the pulse front is plotted as a function of Δ.

Figure 2 shows the dynamics of the delayed system with the history initialized to pulse solutions with different speeds Δ and iteration numbers n.

Pulses with negative values of Δ are not induced through causal relation to the state at $x(t - \tau)$. These pulses have more of a phase wave nature and can never be stable. Nevertheless, they contribute to the properties of the dynamical system.

By perturbing more than one lattice sites of the initial condition, leads to multi pulse solutions. These solutions are similar to multi-bump pulses in spatially extended systems.

Stability — To numerically calculate the stability of the pulse solutions, we cast the delayed system into the form of Eq. (4) and calculate the monodromy matrix as the product of Jacobians along the orbit. The eigenvalues of Monodromy matrix then correspond to the Floquet multipliers z

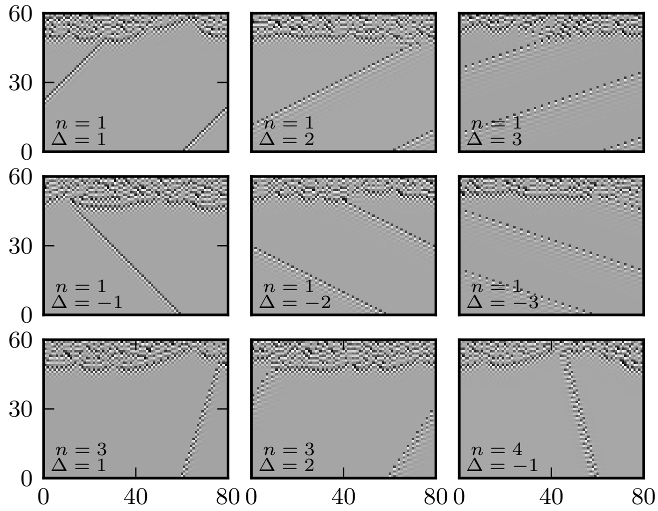

Figure 2: Exemplary space time plots of the delayed dynamics Eq. (14) for initial conditions corresponding to pulse solutions with speed Δ and iteration number n. The plots show 60 iterations of the initial history. Color scale: 0 to 1 $\triangleq$ dark to light. Parameters: $\tau = 81$, $\epsilon = 0.12$, $r = 3.95$.

in Eq. (8).

From the numerical calculations we have made the following observations for the pulse solutions, when the delay time is large: (i) The eigenvalues lie in the complex plane and on a closed curve almost equidistantly spaced outside the unit circle. (ii) The number of eigenvalues on the curve is given (of course) by the period $T = \tau + \Delta$. The curve on which the solutions lie, however, is independent of τ if τ is large enough. (iii) For multi-pulse solutions the eigenvalue curve is almost the same, even if there are many pulses closely packed over the whole period T. (iv) For $\Delta = \pm 1, \pm 2, \pm 3, \ldots$ the eigenvalue curve has 1, 2, 3 loops outside the unit circle. Additional loops lie further away from the origin and thus orbits with large $|\Delta|$ are more unstable.

4. Discussion

Based on our numerical analysis and the analytic approaches above, we conjecture the following for the chaotic map with large delay:

The invariant measure is concentrated on two sets of periodic orbits (these sets of orbits contribute most to the measure): (i) periodic orbits with small periods $T \ll \tau$, and (ii) pulse like orbits with periods close to a multiple of the delay $T = n\tau + \Delta$.

We have shown that the weights of the orbits from the first set remains the same when τ is increased. For the second set of orbits, each individual orbit becomes more unstable when τ is increased, however, the number of periodic orbits in this set also increases with increasing τ: there is a

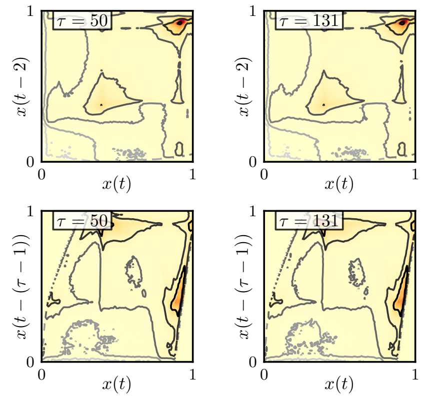

Figure 3: Joint probability distributions for different combination of variables and different values of the delay time. The color code and contour lines indicate the height of the joint probability distribution (in logarithmic scale) and show very characteristic fingerprints of the dynamics.

longer domain to fit in more pulses. In fact, to estimate the number of pulses, let us assume that we can stack pulses together with just one site distance between them. Then can then estimate the number of pulse solutions by

$$N = \sum_{n=1}^{\tau} \frac{1}{n!\tau} \prod_{l=0}^{n-1} (\tau - l) = \frac{2^\tau - 1}{\tau} . \qquad (15)$$

Here, the term under the sum is the number of n-pulse solutions accounting for the fact that all pulses are identical and eliminating the symmetry due to shifting the entire history. We conjecture that the increasing number of orbits of this type matches the decreasing weight that each orbit has.

Our conjecture is motivated by the following observation: The natural measure $\mu(S)$ determines the probability for the trajectory to visit a set S in phase space. In the chaotic regime, we numerically observe that the probability distribution of sequences $(x(t), x(t-1), \ldots, x(t-n))$ with length $n \ll \tau$ is independent of τ. This independence is depicted in the upper two panels in Fig. 3. The panels show as an example the joint probability distribution $P(x(t); x(t-2))$ for $\tau = 50$ and $\tau = 131$. Theses distributions have very characteristic features but still exactly match for different delays. This is true for the joint probability distributions $P(x(t); x(t-1); \ldots, x(t-n))$ for $n \ll \tau$.

Similarly, we observe that joint probability distributions of the form

$$P[x(t);\ x(t-1);\ \ldots x(t-n);\ x(t-\tau+n_a));\ldots x(t-\tau-n_b)] , \qquad (16)$$

with $n, n_a, n_b \ll \tau$ are independent of τ. This is depicted in the lower two panels of Fig. 3, where the joint probability distribution of $x(t)$ and $x(t-(\tau-1))$ are shown for $\tau = 50$ and $\tau = 131$. In the numerical simulations we find further, that the probability distributions of $x(t)$ and $x(t-k)$ where k is neither small nor close to a multiple of the delay time are completely independent.

Conclusion — In conclusion, we have presented analytical and numerical indications that the natural measure of a delayed map with large delay is concentrated on two different types of periodic orbits: (i) periodic orbits with small periods ($T \ll \tau$) and (ii) periodic orbits with periods close to multiples of the delay time ($T = n\tau + \Delta$, with $|\Delta| \ll \tau$). The latter type of orbits correspond to pulse like solutions.

We conjecture that the natural measure converges in a certain sense as the delay time is increased. This is motivated by the fact that the probability distribution of certain patterns becomes independent of the delay time for large delay. We believe that the number of pulse solution grows such that it exactly compensates the increasing instability of a single pulse.

We hope our investigation is a step in understanding the interesting properties of delayed systems better.

References

[1] T. Erneux, *Applied delay differential equations* (Springer, ADDRESS, 2009).

[2] R. Lang and K. Kobayashi, IEEE J. Quantum Electron. **16**, 347 (1980).

[3] K. Ikeda and K. Matsumoto, Physica D **29**, 223 (1987).

[4] I. Fischer, O. Hess, W. Elsäßer, and E. O. Göbel, Phys. Rev. Lett. **73**, 2188 (1994).

[5] L. Larger and J. M. Dudley, Nature **465**, 41 (2010).

[6] M. C. Mackey and L. Glass, Science **197**, 287 (1977).

[7] S. Lepri, G. Giacomelli, A. Politi, and F. T. Arecchi, Physica D **70**, 235 (1993).

[8] J. D. Farmer, Physica D **4**, 366 (1982).

[9] R. Vicente, J. Dauden, P. Colet, and R. Toral, IEEE J. Quantum Electron. **41**, 541 (2005).

[10] C. Grebogi, E. Ott, and J. A. Yorke, Phys. Rev. A **37**, 1711 (1988).

[11] Y. C. Lai, Y. Nagai, and C. Grebogi, Phys. Rev. Lett. **79**, 649 (1997).

[12] Y.-C. Lai, Phys. Rev. E **56**, 6531 (1997).

[13] V. Flunkert, S. Yanchuk, T. Dahms, and E. Schöll, Phys. Rev. Lett. **105**, 254101 (2010).

[14] B. Sandstede and A. Scheel, Journal of Differential Equations **172**, 134 (2001).