2005 International Symposium on Nonlinear
Theory and its Applications (NOLTA2005)
Bruges, Belgium, October 18-21, 2005

Numerical Method of Proving Existence of Periodic Solution for Nonlinear ODE using Affine Arithmetic and Green's Function Expression

Yuchi Kanzawa† and Shin'ichi Oishi‡

†Department of Communication Engineering, Shibaura Institute of Technology
3-9-14 Shibaura, Minato-ku, Tokyo 108-8568 Japan
‡Department of Computer Science, Waseda University
3-4-1, Okubo, Shinjuku-ku, 169-8555, Japan,
Email: kanz@kanz.ce.shibaura-it.ac.jp, oishi@waseda.jp

Abstract—In this paper, a method is proposed to prove the existence of solutions for nonlinear ordinary equations on an essentially bounded functional space. For that, Affine Arithmetic extended to the functional space. Green's Function Expression is also used to prevent from overestimating integral arithmetic with Affine Arithmetic.

1. Introduction

Many methods have been proposed to prove the existence of solutions for nonlinear ordinary differential equations. Especially, one of the authors, Oishi, proposed a method to prove the existence of solutions for nonlinear ordinary differential equations on a continuous functional space. This method uses Krawczyk's operator on a functional space [1]. Krawczyk-like operator is constructed from Newton operator using Mean Value Theorem. This operates from an interval on a functional space to an interval on a functional space. In order to calculate the image of Krawczyk-like operator, Interval Arithmetic on the functional space is used.

In this paper, the following 2 things are revised from [1]. One is that Affine Arithmetic on functional space is used instead of Interval Arithmetic in order to obtain more accurate result of set arithmetic except for integral operation. The other is that the inverse of the linearized operator, which is needed to evaluate Krawczyk-like operator, is reformed in order to obtain more accurate result of integral operation.

2. Preliminaries

In this section, we introduce the theorem to prove the existence of the solution for nonlinear ordinary differential equations.

2.1. Formalization to Operator Equation

In this subsection, we formalize nonlinear ordinary equation to an operator equation on a Banach space.

We consider the following system of first order real differential equations:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(x(t), t), \tag{1}$$

where x is a n-dimensional vector valued function on $[0, 1]$, $f(x(t), t)$ is a n-dimensional vector valued nonlinear function.

In the following, we assume that an approximation of the periodic solution $c(t)$ is given for Eq. (1). We also assume that it is a step function. Under these assumptions, we will present a sufficient condition on which the problem has an exact solution in a domain containing an approximate solution $c(t)$.

Let X be the space of real valued essentially bounded function on the interval $[0, 1]$, let be $x(t) = (x_1(t), x_2(t), \cdots, x_n(t)) \in X^n$, and let X^n be the Banach space with the maximum-supremum norm

$$\|x\|_\infty = \max_{1\leqq i\leqq n} \sup_{t\in[0,1]} |x_i(t)|. \tag{2}$$

In the following, $\|\cdot\|$ means $\|\cdot\|_\infty$. Let $Y = X^n \times R^n$ be the Banach space with the norm

$$\|y\|_Y = \max(\|u\|, \|v\|) \ \text{ for } y = (u, v) \in Y. \tag{3}$$

Let D be a subset of X^n. In the following, vectors and matrices mean n-dimensional vectors and $n \times n$-matrices, respectively. We assume that the given approximate solution $c(t)$ is an element of X^n. We now define an operator $F : D \subset X^n \to Y$ by

$$Fx = \left(\frac{\mathrm{d}x}{\mathrm{d}t} - f(x, t), x(1) - x(0) \right). \tag{4}$$

Then we can rewrite the original problem as the following operator equation:

$$Fx = 0. \tag{5}$$

2.2. Existence Theorem of Solution for Eq.(5)

In this subsection, we shall introduce the existence theorem of the periodic solution for Eq.(5).

In the following, we assume that $f : X^n \to X^n$ is continuously Fréchet differentiable with respect to x. The Jacobian matrix of f with respect to x is denoted by $f_x(x)$. Then it is easy to see that $F : X^n \to Y$ is Fréchet differentiable for an element x of D and the Fréchet derivative $F_x(x) : D \to Y$ is given as follows:

$$F_x(x)h = \left(\frac{\mathrm{d}h}{\mathrm{d}t} - f_x(x)h, h(1) - h(0) \right), \tag{6}$$

where $h \in D$.

For a real matrix valued step function $A(t)$ on $[0, 1]$ which approximate $f_x(x)$, we define the following linear operator

$$Lh = \left(\frac{\mathrm{d}h}{\mathrm{d}t} - A(t)h, h(1) - h(0) \right). \tag{7}$$

Let $\Phi(t)$ be the fundamental matrix of the linear homogeneous differential system

$$\frac{\mathrm{d}\Phi(t)}{\mathrm{d}t} = A(t)\Phi(t) \tag{8}$$

satisfying $\Phi(0) = E$, where E is the unit matrix. Remark that $\Phi(t)$ and $\Phi^{-1}(t)$ can be obtained with guanranteed accuracy using some method [3], [5].

Now we assume that L is invertible. We consider a Newton-like operator $k : X^n \to X^n$

$$\begin{aligned} k(x(t)) &= x(t) - L^{-1}F(x(t) \\ &= L^{-1}\left(Lx(t) - F(x(t))\right) \\ &= L^{-1}\left(\left(\left(\frac{\mathrm{d}x(t)}{\mathrm{d}t} - A(t)x(t)\right) - \left(\frac{\mathrm{d}x(t)}{\mathrm{d}t} - f\left(x(t), t\right)\right),\right.\right. \\ &\qquad \left.(x(1) - x(0)) - (x(1) - x(0))\right) \\ &= L^{-1}\left(f(x(t), t) - A(t)x(t), 0\right). \end{aligned} \tag{9}$$

We now introduce the following theorem.

Theorem 2.1 For a set $U \subset X^n$, if

$$\mathrm{co}\{k(x(t)) \mid x(t) \in U\} \subset U \tag{10}$$

and

$$\max_{x \in U} \|k'(x(t))\| < 1 \tag{11}$$

hold, there is a fixed point x^* of k uniquely in U. □

This theorem is proved by Banach's contraction mapping theorem.

If we can calculate (10) and (11) by computers, we can check whether the periodic solution of Eq. (1) exists or not by computers.

3. Functional Affine Arithmetic

In this section, we introduce functional Affine Arithmetic, which is extended from Affine Arithmetic [2].

Definition 3.1 Let be $c(t), d_i(t) \in X$ $(i \in \{1, 2, \cdots, n\})$, $-1 \leqq \varepsilon_i$ $(i \in \{1, 2, \cdots, n+1\}) \leqq 1$ and $\delta \in \mathrm{R}_{+,0}$. Then the set of functions $\{c(t) + \sum_{i=1}^{n} d_i(t)\varepsilon_i + \delta\varepsilon_{n+1}\} \subset X$ is called functional Affine Form on X. The set of functional Affine Form is denoted by $\mathcal{A}(X)$. □

Definition 3.2 For $a^{(1)}(t), a^{(2)}(t) \in \mathcal{A}(X)$, operations

$$\{a^{(1)}(t) * a^{(2)}(t) | * \in \{+, -, \times, /\}\}$$

and

$$\{\phi(a^{(1)}(t)) | \phi \in \{\sin, \cos, \tan, \exp, \log, \cdots\}\}$$

are determined as

- the result is also an functional Affine Form function
- the set described by the result holds

  $$a^{(1)}(t) * a^{(2)}(t) \supset \{x^{(1)}(t) * x^{(2)}(t) | x^{(1)}(t) \in a^{(1)}(t), x^{(2)}(t) \in a^{(2)}(t)\}$$

  and

  $$\phi(a^{(1)}(t)) \supset \{\phi(x^{(1)}(t)) | x^{(1)}(t) \in a^{(1)}(t)\},$$

  respectively.

Let $a^{(1)}(t)$ and $a^{(2)}(t)$ be

$$a^{(j)}(t) = c^{(j)}(t) + \sum_{i=1}^{n} d_i^{(j)}(t)\varepsilon_i + \delta^{(j)}\varepsilon_{n+1} \quad (j \in \{1, 2\}).$$

Maximum norm of $a^{(1)}(t)$ is overestimated as

$$\|a^{(1)}(t)\|_\infty \leqq \|c^{(1)}(t)\|_\infty + \sum_{i=1}^{n} \|d_i^{(1)}(t)\|_\infty + \delta^{(1)}.$$

Addition and subtraction between $a^{(1)}(t)$ and $a^{(2)}(t)$ are operated as

$$\begin{aligned} a^{(1)}(t) \pm a^{(2)}(t) = c^{(1)}(t) \pm c^{(2)}(t) + \\ \sum_{i=1}^{n}(d_i^{(1)}(t) \pm d_i^{(2)}(t))\varepsilon_i + (\delta^{(1)} + \delta^{(2)})\varepsilon_{n+1} \end{aligned}$$

Addition and subtraction between $a^{(1)}(t)$ and a function $b(t) \in X$ are operated as

$$a^{(1)}(t) \pm b(t) = c^{(1)}(t) \pm b(t) + \sum_{i=1}^{n} d_i^{(1)}(t)\varepsilon_i + \delta^{(1)}\varepsilon_{n+1}.$$

Multiplication between $a^{(1)}(t)$ and $a^{(2)}(t)$ is operated as

$$\begin{aligned} &a^{(1)}(t) \times a^{(2)}(t) = c^{(1)}(t)c^{(2)}(t) \\ &+ \sum_{i=1}^{n}\left(c^{(2)}(t)d_i^{(1)}(t) + c^{(1)}(t)d_i^{(2)}(t)\right)\varepsilon_i \\ &+ \left(\frac{1}{2}\sum_{i=1}^{n}\left\|d_i^{(1)}(t)d_i^{(2)}(t)\right\|_\infty\right. \\ &+ \sum_{j=1, i \neq j}^{n}\sum_{i=1}^{n}\left\|d_i^{(1)}(t)d_j^{(2)}(t) + d_j^{(1)}(t)d_i^{(2)}(t)\right\|_\infty \\ &+ \left.\left\|a^{(1)}(t)\right\|_\infty\delta^{(2)} + \left\|a^{(2)}(t)\right\|_\infty\delta^{(1)}\right)\varepsilon_{n+1} \end{aligned}$$

Multiplication between $a^{(1)}(t)$ and a function $b(t) \in X$ is operated as

$$a^{(1)}(t) \times b(t) = c^{(1)}(t)b(t) + \sum_{i=1}^{n} b(t)d_i^{(1)}(t)\varepsilon_i + \|b(t)\|_\infty \delta^{(1)} \varepsilon_{n+1}.$$

Integration of $a^{(1)}(t)$ from 0 to t is overestimated as

$$\int_0^t a^{(1)}(s)\mathrm{d}s = \int_0^t c^{(1)}(s)\mathrm{d}s + \left(\sum_{i=1}^{n} \|d_i^{(1)}(s)\| + \delta^{(1)}\right)\varepsilon_{n+1},$$

□

Let U be an functional Affine Form $U(t)$. Then the lefthandside of conditions (10) and (11) are evaluated as

$$\begin{aligned} &\mathrm{co}\{k(x(t))|x(t) \in U\} \\ &\subset k(U(t)) \\ &= L^{-1}(f(U(t), t) - A(t)U(t), 0) \end{aligned}$$

and

$$\begin{aligned} &\max_{x(t)\in U} \|k'(x(t))\|_\infty \\ &\leqq \|k'(U(t))\|_\infty \\ &= \|L^{-1}(f'(U(t), t) - A(t), 0)\|_\infty \\ &\leqq \|L^{-1}(f'(U(t), t) - A(t), 0)B(t)\|_\infty, \end{aligned}$$

respectively, where $B(t)$ is a functional Affine Form describing the unit ball

$$B(t) = (\varepsilon_{n+2}, \cdots, \varepsilon_{2n+1})^{\mathrm{tr}}.$$

From this, we can revise conditions (10) and (11) as

$$L^{-1}(f(U(t), t) - A(t)U(t), 0) \subset U(t) \tag{12}$$

and

$$\|L^{-1}(f'(U(t), t) - A(t), 0)B(t)\|_\infty < 1, \tag{13}$$

respectively. If L^{-1} can be expressed explicitly, conditions (12) and (13) can be checked.

4. Evaluating L^{-1}

$L^{-1}(u, v)$ is the solution η of

$$L\eta = (u, v)$$

which is equivalent to

$$\begin{aligned} \frac{\mathrm{d}\eta(t)}{\mathrm{d}t} &= A(t)\eta(t) + u(t), \\ \eta(1) - \eta(0) &= v. \end{aligned}$$

Let $G(s, t)$ be Green's Function of L, that is

$$\begin{aligned} \frac{\mathrm{d}G(s, t)}{\mathrm{d}t} &= A(t)G(s, t) - \Delta(s, t), \\ G(s, 1) - G(s, 0) &= 0, \end{aligned}$$

where

$$\Delta(s, t) = \begin{pmatrix} \delta(s-t) & & 0 \\ & \ddots & \\ 0 & & \delta(s-t) \end{pmatrix}.$$

If $E - Phi(1)$ is invertible, $G(s, t)$ exists and we have

$$G(s, t) = \Phi(t)\left((E - \Phi(1))^{-1} - H(s, t)\right)\Phi(s),$$

where

$$H(s, t) = \begin{cases} 0 & (0 \leqq s \leqq t), \\ E & (t \leqq s \leqq 1). \end{cases}$$

Using $G(s, t)$, $L^{-1}(u, v)$ is described as

$$L^{-1}(u, v) = \int_0^1 G(s, t)u(s)\mathrm{d}s + \Phi(t)(\Phi(1) - E)^{-1}v. \tag{14}$$

From this, we can revise conditions (12) and (13) as

$$\int_0^1 G(s, t)\left(f(U(s), t) - A(s)U(s), 0\right)\mathrm{d}s \subset U(t) \tag{15}$$

and

$$\left\|\int_0^1 G(s, t)\left(f'\left(U(s), s\right) - A(s), 0\right)B(s)\mathrm{d}s\right\|_\infty < 1, \tag{16}$$

respectively.

5. Inclusion of $H(s - t)$

In this section, we obtain an inclusion of $H(s - t)$. $H(s, t)$ is described also as

$$H(s, t) = \begin{pmatrix} h(s, t) & & 0 \\ & \ddots & \\ 0 & & h(s, t) \end{pmatrix},$$

where $h(s, t)$ is Heaviside's step function shown as

$$h(s - t) = \begin{cases} 0 & (0 \leqq s \leqq t), \\ 1 & (t \leqq s \leqq 1). \end{cases}$$

$h(s, t)$ is included by $[\underline{h}(s, t), \bar{h}(s, t)]$, where

$$\begin{aligned} \underline{h}(s, t) &= \frac{1}{2}\left(\frac{b\exp(a(s-t)) + b - 1}{b\exp(a(s-t)) - b + 1} + 1\right) - b, \\ \bar{h}(s, t) &= \frac{1}{2}\left(\frac{(1-b)\exp(a(s-t)) - b}{(1-b)\exp(a(s-t)) + b} + 1\right) + b, \end{aligned}$$

$b > 0$ and $a > 0$. The integral of the width of this interval diminishes for sufficient large a and for sufficient small b, that is,

$$\int_0^1 \bar{h}(s,t) - \underline{h}(s,t) \mathrm{d}s \to 0,$$
$$(a \to +\infty, b \to +0).$$

$\underline{h}(s,t)$ and $\bar{h}(s,t)$ can be obtained as the solution of equations

$$\frac{\mathrm{d}^2\bar{h}(s,t)}{\mathrm{d}s^2} + 2a(\bar{h}(s,t) - (b+0.5))\frac{\mathrm{d}\bar{h}(s,t)}{\mathrm{d}s} = 0,$$
$$\begin{cases} \bar{h}(0,t) &= \frac{1}{2}\left(\frac{(1-b)\exp(-at)-b}{(1-b)\exp(-at)+b} + 1\right) + b, \\ \left.\frac{\mathrm{d}\bar{h}(s,t)}{\mathrm{d}s}\right|_{s=0} &= \frac{ab(1-b)\exp(-at)}{((1-b)\exp(-at)+b)^2} \end{cases}$$

and

$$\frac{\mathrm{d}^2\underline{h}(s,t)}{\mathrm{d}s^2} + 2a(\underline{h}(s,t) - (b+0.5))\frac{\mathrm{d}\underline{h}(s,t)}{\mathrm{d}s} = 0,$$
$$\begin{cases} \underline{h}(0,t) &= \frac{1}{2}\left(\frac{b\exp(-at)+b-1}{b\exp(-at)-b+1} + 1\right) - b, \\ \left.\frac{\mathrm{d}\underline{h}(s,t)}{\mathrm{d}s}\right|_{s=0} &= \frac{ab(1-b)\exp(-at)}{(b\exp(-at)-b+1)^2}, \end{cases}$$

respectively. Therefore, the inclusion of $H(s,t)$ can be obtained by solving these equations using some numerical validation method [3], [5].

6. Numerical Verification

In this section, an algorithm is shown in order to verify the unique existence of the solution of Eq.(1) from a given approximate solution.

Algorithm 6.1 For a given approximate solution $c(t)$, our numerical verification algorithm is as below:

1. Obtain $A(t)$ from $c(t)$ and $f(x(t),t)$ using Automatic Differentiation.
2. Obtain inclusions of $\Phi(t)$,$\Phi^{-1}(t)$ $H(s,t)$ using some methods
3. If $E-\Phi(1)$ is invertible, go to the next step. Otherwise, the existence test fails.
4. Calculate $f(c(t),t) - A(t)c(t)$ using piecewise machine Interval Arithmetic.
5. Obtain the inclusion of $k(c(t))$ from $\Phi(t)$, $\Phi^{-1}(t)$, $H(s,t)$ and $f(c(t),t) - A(t)c(t)$, using (9) and (14). [3],[5].
6. Set an functional Affine Form $U(t)$ as
$$U(t) = c(t) + \sum_{i=1}^{n} 2\|k(c(t)) - c(t)\|_\infty \varepsilon_i$$
7. Check conditions (15) and (16) using functional Affine Arithmetic. If these conditions hold, then we can find a unique solution of (1) in $U(t)$.

□

7. Numerical Example

In this section, a simple numerical example is shown.

Let us consider the equation described as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -2\pi x_2 - 4\pi^2 x_1^3 + 0.6\pi^2\cos(2\pi t) \end{pmatrix}, \quad (17)$$

We obtained an approximation $c(t)$ of periodic solution of (17) as the 19th ordered power polynomial. By Algorithm 6.1, We obtain a region U

$$U = c(t) + 1.863 \times 10^{-3} B$$

and found a unique solution for Eq.(17) in U.

Acknowledgment

The authors would like to thank Prof. Masahide Kashiwagi of Waseda University for their valuable discussions.

References

[1] Oishi, S.: "Two Topics in Nonlinear System Analysis through Fixed Point Theorems", IEICE Trans. Fundamentals. Vol.E77-A No.7, pp.1144–1153 (1994).

[2] Marcus Vinicius A. Andrade, Joao L. D. Comba and Jorge Stolfi: "Affine Arithmetic", INTERVAL'94, St. Petersburg (Russia), March 5-10, 1994.

[3] Lohner, R. J.: "Enclosing the Solutions of Ordinary Initial and Boundary Value Problems", Computer arithmetic (eds. Kaucher, E. et al.), B. G. Teubner, Stuttgart, pp.255-286 (1987).

[4] Kanzawa, Y. and Oishi, S.: "A method to Prove Existence of Solutions for Nonlinear ODE on Essentially Bounded Functional Space", Proc. 2002 International Symposium on Nonlinear Theory and Its Applications, pp.126–130 (1994).

[5] Kashiwagi, M: 'Numerical Validation for Ordinary Differential Equations Using Power Series Arithmetic", Proc. NOLTA'94, pp.213–218 (1994).