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A Computer-Assisted Existence and Multiplicity Proof for Travelling Waves in a Nonlinearly Supported Beam

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Abstract—We consider a nonlinear fourth-order ordinary differential equation on the whole real line, which models travelling waves in a nonlinearly supported beam, e.g. in a suspension bridge. Our aim is to prove that this problem has at least 36 solutions, for a fixed chosen value of the wave speed parameter.

Our proof makes heavy use of computer assistance: Starting from numerical approximations, we use a fixed-point argument to prove existence of solutions "close to" the computed approximations. The main subtask to be accomplished in this argument is an examination of the spectra of the operators arising by linearization at the computed approximations.

1. Introduction

In the mathematical investigation of oscillations of suspension bridges, the nonlinear beam equation

$$u^{iv} + c^2 u'' + e^u - 1 = 0 \quad \text{on} \quad \mathbb{R} \tag{1}$$

arises from a travelling wave ansatz $u(x - ct)$ for a corresponding wave equation; see [2, 3].

Until recently, there has been little progress on the proof of existence of solutions of equation (1), until Smets and van den Berg [6] showed that for *almost all c* in the interval $(0, \sqrt{2})$, there exists at least one solution.

Here we go in a different direction to prove existence of many homoclinic solutions of equation (1) for one fixed c, also assuming that $c \in (0, \sqrt{2})$. We first calculate approximate solutions numerically. The next step is to verify that there are true solutions of (1) close to each of the approximate solutions. This is done by a fixed point argument applied to the differential equation for the error function. The result is:

Theorem 1 *For $c = 1.3$, equation (1) has at least 36 solutions.*

For preparing our computer-assisted proof of Theorem 1, let $H^2_S(\mathbb{R}) := \{u \in H^2(\mathbb{R}) : u(x) = u(-x) \text{ for all } x \in \mathbb{R}\}$, endowed with the inner product $\langle u, v\rangle_{H^2} := \langle u'', v''\rangle_{L^2} + \sigma\langle u, v\rangle_{L^2}$, where $\langle\cdot,\cdot\rangle_{L^2}$ denotes the usual inner product in $L^2(\mathbb{R})$, and $\sigma > 0$ is some constant to be specified later.

Besides $H^2_S(\mathbb{R})$, we will need its (topological) dual space $H^{-2}_S(\mathbb{R})$, endowed with the canonical dual norm $\|\cdot\|_{H^{-2}}$.

2. The Existence and Enclosure Theorem

The basis of our computational existence and multiplicity proof for problem (1) is the following existence and enclosure Theorem 2. Besides existence of a solution $u^* \in H^2_S(\mathbb{R})$, the theorem yields a *bound* for u^* of the form

$$\|u^* - \omega\|_{H^2} \le \alpha, \tag{2}$$

with $\omega \in H^2_S(\mathbb{R})$ denoting an approximate solution computed by numerical means, and with $\alpha > 0$ denoting a "small" constant provided by the theorem. Thus, with $\omega_1, \ldots, \omega_k \in H^2_S(\mathbb{R})$ denoting approximations such that, with $\alpha_1, \ldots, \alpha_k$ denoting the error bounds given by the theorem,

$$\|\omega_i - \omega_j\|_{H^2} > \alpha_i + \alpha_j \tag{3}$$

for $i, j = 1, \ldots, k,\ i \ne j$, our method yields the existence of k different solutions $u^*_1, \ldots, u^*_k \in H^2_S(\mathbb{R})$ and thus, the desired multiplicity result. Note that (3) can be checked rather directly from the numerical data.

So let $\omega \in H^2_S(\mathbb{R})$ denote an approximate solution to problem (1) obtained by numerical means. We need the following two quantities:

(i) a bound $\delta \ge 0$ for the *defect* (residual) of ω:

$$\|\omega^{iv} + c^2\omega'' + e^\omega - 1\|_{H^{-2}} \le \delta, \tag{4}$$

(ii) a constant $K \ge 0$ such that

$$\|u\|_{H^2} \le K\, \|Lu\|_{H^{-2}} \ \text{ for all } \ u \in H^2_S(\mathbb{R}), \tag{5}$$

with $L : H^2_S(\mathbb{R}) \to H^{-2}_S(\mathbb{R})$ denoting the linearization of (1) at ω:

$$Lu := u^{iv} + c^2u'' + e^\omega u, \ \text{ i.e.} \quad (Lu)[\varphi] = \int_{\mathbb{R}} (u''\varphi'' - c^2u'\varphi' + e^\omega u\varphi)dx. \tag{6}$$

Furthermore, let $\bar\omega := \sup_{x\in\mathbb{R}} \omega(x)$, and $\widehat{C} := \frac{1}{2}\left(\frac{3}{\sigma}\right)^{3/8}$, which can be shown to satisfy the embedding inequality $\|u\|_\infty \le \widehat{C}\|u\|_{H^2}$ for all $u \in H^2_S(\Omega)$.

Theorem 2 *Suppose that some $\alpha \geq 0$ exists such that*

$$\delta \leq \frac{\alpha}{K} - \alpha^2 \frac{\widehat{C}}{2\sigma} \exp(\bar{\omega} + \widehat{C}\alpha) \tag{7}$$

and

$$\alpha K \frac{\widehat{C}}{\sigma} \exp(\bar{\omega} + \widehat{C}\alpha) < 1. \tag{8}$$

Then, there exists a solution $u^* \in H_S^2(\mathbb{R})$ of problem (1) satisfying (2).

Sketch of Proof: The first step is to note that $L : H_S^2(\mathbb{R}) \to H_S^{-2}(\mathbb{R})$ is one-to-one and onto. Via the transformation $v = u - \omega$ problem (1) is therefore equivalent to

$$v = -L^{-1}\left[e^{\omega}(e^{v} - 1 - v) + (\omega^{iv} + c^2\omega'' + e^{\omega} - 1)\right] =: Tv,$$

which amounts to a fixed-point equation for $T : H_S^2(\mathbb{R}) \to H_S^2(\mathbb{R})$. Let $\mathcal{D} := \{v \in H_S^2(\mathbb{R}) : \|v\|_{H^2} \leq \alpha\}$, with α satisfying (7) and (8). (4), (5), (7) and (8) imply that $T(\mathcal{D}) \subset \mathcal{D}$ and that T is a contraction on $\mathcal{D}$, whence Banach's Fixed Point Theorem gives a fixed point $v^* \in \mathcal{D}$ of T, i.e. $u^* := \omega + v^*$ is a solution of (1) satisfying (2).

3. Computation of K

We give a brief description how a constant K satisfying (5) can be computed explicitly, as needed for Theorem 2. We use analytical as well as additional computer-assisted arguments.

With $\Phi : H_S^2(\mathbb{R}) \to H_S^{-2}(\mathbb{R})$ denoting the canonical isometric isomorphism, we note that

$$\|Lu\|_{H^{-2}} = \|\Phi^{-1}Lu\|_{H^2} \text{ for } u \in H_S^2(\mathbb{R}), \tag{9}$$

and that, by (6),

$$\langle \Phi^{-1}Lu, v\rangle_{H^2} = (Lu)[v] = \int_{\mathbb{R}} (u''v'' - c^2u'v' + e^{\omega}uv)dx$$

for $u, v \in H_S^2(\mathbb{R})$, which in particular implies that $\Phi^{-1}L$ is $\langle\cdot,\cdot\rangle_{H^2}$-symmetric. Since $\Phi^{-1}L$ is moreover defined on the whole of $H_S^2(\mathbb{R})$, it is therefore selfadjoint (and bounded). Thus, using (9) and the spectral decomposition of $\Phi^{-1}L$, we see that (5) holds if and only if

$$\gamma := \min\{|\lambda| : \lambda \text{ is in the spectrum of } \Phi^{-1}L\} > 0, \tag{10}$$

and that in the affirmative case one can choose any $K \geq \frac{1}{\gamma}$.

Thus, we have to compute a positive lower bound for γ (proving simultaneously that (10) holds true). The first step is to calculate the *essential* spectrum σ_{ess} of $\Phi^{-1}L$ (defined as the set of all accumulation points of the spectrum, i.e. the spectrum except isolated eigenvalues; note that eigenvalues of infinite multiplicity cannot occur for our ODE problem). For technical simplification, we will now assume that *the approximate solution ω has compact support.* Using compact perturbation arguments, and Fourier transform methods, we obtain that

$$\sigma_{\text{ess}} = \left[\frac{1}{2}\left(1 + \frac{1}{\sigma}\right) - \sqrt{\frac{1}{4}\left(1 - \frac{1}{\sigma}\right)^2 + \frac{c^4}{4\sigma}}, \max\left\{1, \frac{1}{\sigma}\right\}\right].$$

Since besides σ_{ess} only isolated eigenvalues of $\Phi^{-1}L$ contribute to its spectrum, we are left to compute a positive lower bound for

$$\gamma_0 := \min\{|\lambda| : \lambda \text{ is isolated eigenvalue of } \Phi^{-1}L\},$$

the computation of which needs *eigenvalue bounds* obtained by computer-assisted means of their own, which we will not describe here. See [1, 4, 5] for details.

4. Numerical results

A large number of numerical solutions to problem (1) was found using a shooting method. Starting from 40 computed shooting approximations, we applied a Newton-collocation method to improve the quality of the approximations. In all 40 cases, the Newton iteration "converged" within about 6 steps, with a tolerance of 10^{-7}, to highly accurate approximations ω. By the methods described above we were able to compute the constants K satisfying (5).

The results are displayed in Table 1, as well as the computed defect bounds δ, and the error bounds α provided by Theorem 2; the crucial conditions (7) and (8) are satisfied in 36 of the 40 cases. In the remaining 4 cases, the constant K is too large, and no α satisfying (7) and (8) could be found for the values of δ obtained within our approximation quality.

Finally, it is easy to check that condition (3) holds true. This completes the desired existence and multiplicity result, i.e. the proof of Theorem 1.

References

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Table 1: Verified upper bounds for the crucial constants K, α, δ.

	lower branch				upper branch			
Solution	K	δ	α	Morse Index	K	δ	α	Morse Index
1	1.51e+01	5.36e-08	8.05e-07	1	2.48e+01	4.21e-08	1.05e-06	1
2	6.52e+01	4.56e-08	2.97e-06	2	1.27e+02	4.40e-08	5.59e-06	3
3	1.22e+02	2.06e-08	2.50e-06	1	6.21e+01	4.62e-08	2.87e-06	2
4	3.61e+02	4.87e-08	1.76e-05	2	8.55e+02	4.41e-08	3.80e-05	3
5	8.06e+02	5.32e-08	4.33e-05	1	1.09e+02	4.02e-08	4.37e-06	2
6	2.11e+03	5.18e-08	1.18e-04	2	5.24e+03	6.53e-11	3.42e-07	3
7	5.11e+03	4.70e-08	4.33e-05	1	3.48e+02	4.33e-08	1.51e-05	2
8	3.19e+04	1.13e-10	3.72e-06	1	2.07e+03	1.62e-10	3.34e-07	2
9	7.87e+04	1.57e-10	–	-	1.99e+05	5.37e-10	–	-
10	3.19e+04	1.57e-10	5.21e-06	1	1.30e+04	2.62e-10	3.44e-06	2
11	1.87e+06	7.69e-11	–	-	8.12e+04	3.08e-10	–	-
12	9.20e+01	5.18e-08	4.77e-06	2	1.14e+02	2.65e-08	3.02e-06	3
13	1.20e+02	4.69e-08	5.62e-06	3	2.35e+02	4.40e-08	1.04e-05	4
14	2.65e+02	2.03e-08	5.35e-06	2	1.65e+02	4.47e-08	7.35e-06	3
15	7.00e+02	5.25e-08	3.71e-05	3	1.56e+03	1.67e-08	2.61e-05	4
16	3.80e+02	4.85e-08	1.85e-05	2	2.32e+02	4.62e-08	1.07e-05	3
17	1.45e+02	4.97e-08	7.16e-06	3	2.23e+02	1.65e-08	3.65e-06	4
18	1.97e+02	2.11e-08	4.16e-06	4	3.70e+02	1.73e-08	6.38e-06	5
19	4.12e+03	5.50e-08	4.16e-06	4	6.81e+03	3.34e-09	2.37e-05	5
20	2.43e+03	7.36e-08	2.02e-04	4	2.17e+02	6.38e-10	3.34e-07	5

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