2005 International Symposium on Nonlinear Theory and its Applications (NOLTA2005)
Bruges, Belgium, October 18-21, 2005

Some computer assisted proofs on the bifurcation structure of solutions for heat convection problems

M. T. Nakao*, Y. Watanabe**, N. Yamamoto† T. Nishida‡ and M-N. Kim*

*Faculty of Mathematics, Kyushu University
**Computing and Communications Center, Kyushu University
†Department of Computer Science and Information Mathematics, The University of Electro-Communications
‡Department of Mathematical Sciences, Waseda University
E-mail: mtnakao@math.kyushu-u.ac.jp

Abstract—In this paper, we present several results on cmputer assisted approaches for solutions of the two-dimensional Rayleigh-Benard convection problems. First, we will describe on a basic concept of our numerical verification method to prove the existence of the steady-state solutions based on the infinite dimensional fixed-point theorem using Newton-like operator with the spectral approximation and the constructive error estimates. Next, we show some verification examples of several exact non-trivial solutions for the given Prandtl and Rayleigh numbers. Furthermore, a computer assisted proof of the existence for a symmetry breaking bifurcation point will be presented, which should be an important information to clarify the global bifurcation structure. We will also consider the extension of these results to the three dimensional problems.

1. The Rayleigh-Bénard Problems

We consider a plane horizontal layer (see Fig.1) of an incompressible viscous fluid heated from below. At the lower boundary: $z = 0$ the layer of fluid is maintained at temperature $T + \delta T$ and the temperature of the upper boundary $(z = h)$ is T.

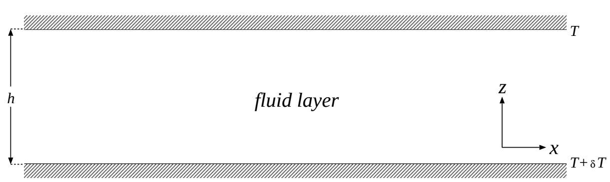

Fig.1. Model of fluid layer

As well known, under the vanishing assumption in y-direction, the two-dimensional (x-z) heat convection model can be described as the following Oberbeck-Boussinesq approximations [1, 3]:

$$\left\{\begin{array}{rcl} u_t + uu_x + wu_z &=& -p_x/\rho_0 + \nu\Delta u, \\ w_t + uw_x + ww_z &=& -(p_z + g\rho)/\rho_0 + \nu\Delta w, \\ u_x + w_z &=& 0, \\ \theta_t + u\theta_x + w\theta_z &=& \kappa\Delta\theta. \end{array}\right. \tag{1}$$

Here,

$u,\ w$: velocity in x and z, respectively
p : pressure
θ : temperature
ρ : fluid density
ρ_0 : density at temperature $T + \delta T$
ν : kinematic viscosity
g : gravitational acceleration
κ : coefficient of thermal diffusivity
$*_\xi := \partial/\partial\xi (\xi = x, z, t)$
$\Delta := \partial^2/\partial x^2 + \partial^2/\partial z^2$.

And ρ is assumed to be represented by

$$\rho - \rho_0 = -\rho_0\alpha(\theta - T - \delta T),$$

where α is the coefficient of thermal expansion.

The Oberbeck-Boussinesq equations (1) have the following stationary solution:

$$u^* = 0, \quad w^* = 0, \quad \theta^* = T + \delta T - \frac{\delta T}{h}z,$$

$$p^* = p_0 - g\rho_0(z + \frac{\alpha\delta T}{2h}z^2),$$

where p_0 is a constant. By setting

$$\hat{u} := u, \qquad \hat{w} := w, \qquad \hat{\theta} := \theta^* - \theta, \qquad \hat{p} := p^* - p,$$

we obtain the transformed equations:

$$\left\{\begin{array}{rcl} \hat{u}_t + \hat{u}\hat{u}_x + \hat{w}\hat{u}_z &=& \hat{p}_x/\rho_0 + \nu\Delta\hat{u}, \\ \hat{w}_t + \hat{u}\hat{w}_x + \hat{w}\hat{w}_z &=& \hat{p}_z/\rho_0 - g\alpha\hat{\theta} + \nu\Delta\hat{w}, \\ \hat{u}_x + \hat{w}_z &=& 0, \\ \hat{\theta}_t + \delta T\hat{w}/h + \hat{u}\hat{\theta}_x + \hat{w}\hat{\theta}_z &=& \kappa\Delta\hat{\theta}. \end{array}\right. \tag{2}$$

By further transforming to dimensionless variables:

$$t \to \kappa t, \quad u \to \hat{u}/\kappa,$$

$$w \to \hat{w}/\kappa, \quad \theta \to \hat{\theta}h/\delta T, \quad p \to \hat{p}/(\rho_0\kappa^2)$$

of (2), we have the dimensionless equations:

$$\left\{\begin{array}{rcl} u_t + uu_x + wu_z &=& p_x + \mathcal{P}\Delta u, \\ w_t + uw_x + ww_z &=& p_z - \mathcal{P}\mathcal{R}\,\theta + \mathcal{P}\Delta w, \\ u_x + w_z &=& 0, \\ \theta_t + w + u\theta_x + w\theta_z &=& \Delta\theta. \end{array}\right. \tag{3}$$

Here

$$\mathcal{R} := \frac{\delta T \alpha g}{\kappa \nu h} \quad \text{Rayleigh number}$$

and

$$\mathcal{P} := \frac{\nu}{\kappa} \quad \text{Prandtl number.}$$

2. Fixed-point formulation of problem

We describe the problem concerned as a fixed point equation of a compact map on the appropriate function space. Since we only consider the the *steady-state solutions*, u_t, w_t and θ_t vanish in (3). And also assume that all fluid motion is confined to the rectangular region $\Omega := \{0 < x < 2\pi/a,\ 0 < z < \pi\}$ for a given wave number $a > 0$.

Let us impose periodic boundary condition (period $2\pi/a$) in the horizontal direction, stress-free boundary conditions ($u_z = w = 0$) for the velocity field and Dirichlet boundary conditions ($\theta = 0$) for the temperature field on the surfaces $z = 0, \pi$, respectively.

Furthermore, we assume the following evenness and oddness conditions:

$$u(x,z) = -u(-x,z), \quad w(x,z) = w(-x,z),$$
$$\theta(x,z) = \theta(-x,z).$$

We use the stream function Ψ satisfying

$$u = -\Psi_z, \quad w = \Psi_x$$

so that $u_x + w_z = 0$. By some simple calculations in (3) with setting $\Theta := \sqrt{\mathcal{PR}}\theta$, we obtain

$$\left\{ \begin{array}{rcl} \mathcal{P}\Delta^2\Psi &=& \sqrt{\mathcal{PR}}\,\Theta_x - \Psi_z\Delta\Psi_x + \Psi_x\Delta\Psi_z, \\ -\Delta\Theta &=& -\sqrt{\mathcal{PR}}\,\Psi_x + \Psi_z\Theta_x - \Psi_x\Theta_z. \end{array} \right. \tag{4}$$

From the boundary conditions, the functions Ψ and Θ can be assumed to have the following representations:

$$\begin{aligned} \Psi &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin(amx)\sin(nz), \\ \Theta &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} B_{mn}\cos(amx)\sin(nz). \end{aligned} \tag{5}$$

We now define the following function spaces for integers $k \geq 0$:

$$X^k := \left\{ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin(amx)\sin(nz) \mid A_{mn} \in \mathbb{R}, \right. \left. \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}((am)^{2k} + n^{2k})A_{mn}^2 < \infty \right\},$$

$$Y^k := \left\{ \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} B_{mn}\cos(amx)\sin(nz) \mid B_{mn} \in \mathbb{R}, \right. \left. \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}((am)^{2k} + n^{2k})B_{mn}^2 < \infty \right\}.$$

In order to get the enclosure of the exact solutions for the problem (4), we transform the concerned equation into the following fixed point form of a compact map F on $X^3 \times Y^1$:

$$w = Fw. \tag{6}$$

Therefore, by the Schauder fixed-point theorem, if we find a nonempty, closed, bounded and convex set $W \subset X^3 \times Y^1$, satisfying

$$FW \subset W \tag{7}$$

then there exists a solution of (6) in W. The set W in (7) is referred as *a candidate set* of solutions. The set W is usually constructed by computer as a direct sum of the finite dimensional subset $W_N \subset X_N^3 \times Y_N^1 \subset X^3 \times Y^1$ and its orthogonal complement $W_N^{\perp}$, where X_N^3 and Y_N^1 are $N-$*truncated* subspace of X^3 and Y^1, respectively. By using an appropriate projection $P_N : X^3 \times Y^1 \longrightarrow X_N^3 \times Y_N^1$, the decomposed form $P_N FW \subset W_N$ and $(I - P_N)FW \subset W_N^{\perp}$ are verified instead of (7), which is a sufficient condition of (6). Furthermore, in general, a kind of Newton-type formulation is utilized so that the concerning operator has the retraction property in a neighborhood of the solution(see, e.g., [6], [8] etc. for details).

3. Verification of bifurcating solutions

By using the Newton-like procedure similar to that in [6], we succeeded to verify various kinds of bifurcating solutions as shown in Fig. 2. Here, $\mathcal{R}_{\mathcal{C}}$ implies the critical Rayleigh number which equals 6.75. The vertical axis stands for the absolute value of the coefficient of the approximate solutin for Θ. And each dot in Fig. 2 means that the the exact solution corresponding to the point was numerically verified.

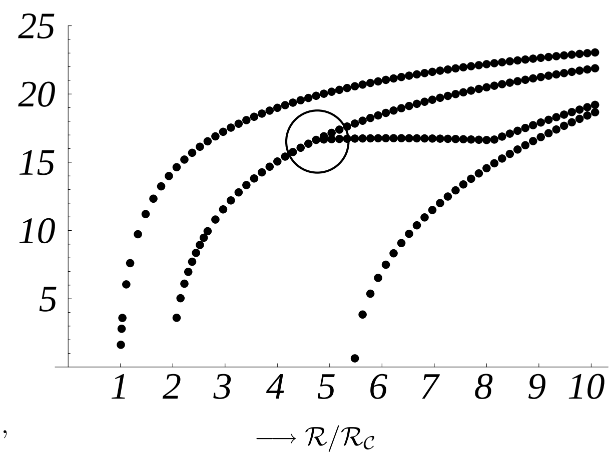

Fig.2. Verified bifurcating solutions

4. Extended System

From the observation of Fig.2, particularly around the part enclosed by the circle, we expected that there should exist some secondary bifurcation. Namely, near "the bifurcation-like point" we found the following two different kninds of approximate solutions. For approximate solutions of the form

$$\Psi_{MN} = \sum_{m=1}^{M}\sum_{n=1}^{N} A_{mn}\sin(amx)\sin(nz),$$
$$\Theta_{MN} = \sum_{m=0}^{M}\sum_{n=1}^{N} B_{mn}\cos(amx)\sin(nz)$$

we have following two solutions satisfying

$$A_{mn} = B_{mn} = 0, \quad m = 1, 3, 5, 7, \ldots \text{with } \mathcal{R} = 32$$

and

$$A_{mn} \neq 0, \quad B_{mn} \neq 0, \quad m = 1, 3, 5, 7, \ldots \text{with } \mathcal{R} = 33.$$

These approximate results strongly suggest that there should exist some symmetry-breaking bifurcation point between $32 \leq \mathcal{R} \leq 33$.

In order to obtain the enclosure of the bifurcation point, we set

$$Z := X^3 \times Y^1, \qquad G := I - F$$

and an operator $S : Z \longrightarrow Z$ by

$$Sw = S(\Psi, \Theta) := (\Psi(x + \pi/a, z), \Theta(x + \pi/a, z))$$

satisfying $SGw = GSw$. Using this "symmetric" operator S, we have the decomposition

$$Z = Z_s \oplus Z_a,$$

where $Z_s = \{w \in Z; Sw = w\}$ and $Z_a = \{w \in Z; Sw = -w\}$. Next, considering $\mathcal{R}$ as a variable, let $\mathcal{G}$ be a map on $Z_s \times Z_a \times \mathbb{R}$ defined by

$$\mathcal{G}(w, v, \mathcal{R}) := \begin{pmatrix} G(w, \mathcal{R}) \\ D_w G[w, \mathcal{R}]v \\ \mathcal{L}(v) - 1. \end{pmatrix} \tag{8}$$

Here $\mathcal{L}$ is an appropriate functional on Z_a. We tried to prove that the extended system $\mathcal{G}(w, v, \mathcal{R}) = 0$ has an isolated solution $(w_0, v_0, \mathcal{R}_0) \in Z_s \times Z_a \times \mathbb{R}$ as well as to verify a sufficient condition such that $\mathcal{R}_0$ is a symmetry-breaking bifurcation point of $G(w, \mathcal{R}) = 0$ by a computer-assisted approach using our verification principle in the section 2.

Using a numerical verification method based on Banach's fixed point theorem(*cf.*[7],[11]), we proved there exists an isolated solution of $\mathcal{G}(w_0, v_0, \mathcal{R}_0) = 0$. Here

$$\mathcal{R}_0 \in 32.04265510708193 + [-9.902, 9.902] \times 10^{-10}.$$

From the bifurcation theorem in [4], it implies that there exists an actual bifurcation point in this interval if

$$D_w G[w_0, \mathcal{R}_0] \ \text{ is invertible on } Z_s, \tag{9}$$

which is a sufficient condition of the existence of a symmetry-breaking bifurcation point. We actually succeeded in the verification of the condition (9) by using a method similar to that an eigenvalue excluding technique in [5]. Thus, it was numerically proved that there exists a symmetry-breaking bifurcation point in the above interval.

5. Three Dimensional Case

For the three dimensional heat convection, more realistic and interesting bifurcation phenomena are observed in the actual problems in fluid mechanics(e.g., [10]). Our verification technique can also be extended to this case. Of course, main difficulty comes from tha fact that we could no longer use the formulation by the stream function. Therefore, we have to apply the verification method directly to the original 3-dimensional Navier-Stokes equation of the form:

$$\left\{ \begin{array}{rcl} \frac{1}{\mathcal{P}}\mathbf{u}\cdot\nabla\mathbf{u} + \nabla p &=& \Delta\mathbf{u} + \mathcal{R}\theta\nabla z, \\ \mathbf{u}\cdot\nabla\theta &=& \Delta\theta + w \\ \nabla\cdot\mathbf{u} &=& 0. \end{array} \right. \tag{10}$$

Here, $\mathbf{u} = (u, v, w)$ and the domain is assumed to be a rectangle such that

$$0 \leq x \leq \frac{2\pi}{a}, \quad 0 \leq y \leq \frac{2\pi}{b}, \quad 0 \leq z \leq \pi,$$

where a, b are constants. Under some appropriate assumptions on the boundary conditions and usual even- or odd-ness coditions for the unknown functions, we look for the solution to (10) of the form, for multi-index $\alpha \equiv (l, m, n)$,

$$\left\{ \begin{array}{rcl} u(x,y,z) &=& \sum_\alpha u_\alpha \sin alx \cos bmy \cos nz, \\ v(x,y,z) &=& \sum_\alpha v_\alpha \cos alx \sin bmy \cos nz, \\ w(x,y,z) &=& \sum_\alpha w_\alpha \cos alx \cos bmy \sin nz, \\ \theta(x,y,z) &=& \sum_\alpha \theta_\alpha \cos alx \cos bmy \sin nz, \\ p(x,y,z) &=& \sum_\alpha p_\alpha \cos alx \cos bmy \cos nz. \end{array} \right. \tag{11}$$

Then the divergence free condition can be written as, for each l, m, n,

$$alu_{lmn} + bmv_{lmn} + nw_{lmn} = 0.$$

Now, for $1 \leq i \leq 4$, we define the functions ϕ_i^α by

$$\phi_1^\alpha \equiv K_0 \sin alx \cos bmy \cos nz,$$

$$\phi_2^\alpha \equiv K_0 \cos alx \sin bmy \cos nz,$$

$$\phi_3^\alpha \equiv K_0 \cos alx \cos bmy \sin nz,$$
$$\phi_4^\alpha \equiv K_0 \cos alx \cos bmy \cos nz,$$

where $K_0 = \dfrac{2\sqrt{2}}{\sqrt{|\Omega|}}$ and $|\Omega|$ is the volume of the domain $(= \dfrac{4\pi^2}{ab})$. Now, setting

$$A_\alpha^2 = (al)^2 + (bm)^2 + n^2 = B_\alpha^2 + n^2,$$

we define the new base vector fields $\{\mathbf{\Phi}^\alpha, \mathbf{\Psi}^\alpha\}$ as follows:

$$\mathbf{\Phi}^\alpha = -\mathbf{e}_1 \frac{aln}{A_\alpha B_\alpha}\phi_1^\alpha - \mathbf{e}_2 \frac{bmn}{A_\alpha B_\alpha}\phi_2^\alpha + \mathbf{e}_3 \frac{B_\alpha}{A_\alpha}\phi_3^\alpha,$$

$$\mathbf{\Psi}^\alpha = \begin{cases} \mathbf{e}_1 \dfrac{bm}{B_\alpha}\phi_1^\alpha - \mathbf{e}_2 \dfrac{al}{B_\alpha}\phi_2^\alpha, & \text{when} \quad l, m \neq 0, \\ \mathbf{e}_2 \dfrac{bm}{A_\alpha}\phi_2^\alpha + \mathbf{e}_3 \dfrac{n}{A_\alpha}\phi_3^\alpha, & \text{when} \quad l = 0, \\ \mathbf{e}_1 \dfrac{al}{A_\alpha}\phi_1^\alpha + \mathbf{e}_3 \dfrac{n}{A_\alpha}\phi_3^\alpha, & \text{when} \quad m = 0. \end{cases}$$

Then it is seen that $\{\mathbf{\Phi}^\alpha, \mathbf{\Psi}^\alpha\}$ constitutes a orthogonal basis of the vector field $\mathcal{V}^k$ which is defined similar to $(H^k(\Omega))^3$ with divergence free condition. And the function space $\mathcal{T}^k$ for temperature is defined as the set of functions represented by the series constituted from $\{\phi_3^\alpha\}$.
Then a solution of the equation (10) can also be formulated as a fixed point of some compact operator on $\mathcal{X} \equiv \mathcal{V}^k \times \mathcal{T}^k$. Thus, by considering the projection to the finite dimensional subspace $\mathcal{X}_n$ of $\mathcal{X}$ as well as the constructicve error estimates for it, we can formulate the verification procedure for the solution of (10) to get enclosure of bifurcating solutions for three dimensional problems.

The computational results will be presented in forthcoming paper.

References

[1] Chandrasekhar, S.: *Hydrodynamic and Hydromagnetic Stability*, Oxford University Press, 1961.

[2] Curry, J. H.: Bounded solutions of finite dimensional approximations to the Boussinesq equations, *SIAM J. Math. Anal.* **10**, pp.71–79 (1979).

[3] Getling, A. V.: *Rayleigh-Bénard Convection: structures and dynamics*, Advanced series in nonlinear dynamics Vol.11, World Scientific, 1998.

[4] Kawanago, T.: A symmetry-breaking bifurcation theorem and some related theorems applicable to maps having unbounded derivatives, Japan Journal of Industrial and Applied Mathematics 21, pp.57-74 (2004).

[5] Nagatou, K., Yamamoto, N. & Nakao, M.T., An approach to the numerical verification of solutions for nonlinear elliptic problems with local uniqueness, Numerical Functional Analysis and Optimization **20** pp. 543-565 (1999).

[6] Nakao, M.T., Numerical verification methods for solutions of ordinary and partial differential equations, Numerical Functional Analysis and Optimization 22(3&4), pp.321–356 (2001).

[7] Nakao, M.T., Hashimoto, K. & Watanabe, Y., A numerical method to verify the invertibility of linear elliptic operators with applications to nonlinear problems, Computing 75, pp.1-14 (2005).

[8] Y. Watanabe, N. Yamamoto, M. T. Nakao and T. Nishida, A Numerical Verification of Nontrivial Solutions for the Heat Convection Problem, *Journal of Mathematical Fluid Mechanics*, Vol.6, No.1, pp.1–20 (2004).

[9] M. T. Nakao, Y. Watanabe, N. Yamamoto and T. Nishida, Some Computer Assisted Proofs for Solutions of the Heat Convection Problems, *Reliable Computing*, Vol.9, No.5, pp.359-372 (2003).

[10] Tsutomu Ikeda, Hideaki Yoshihara and Takaaki Nishida , Pattern formation of heat convection problems, Proc. Internat. Symp. Math. Modeling and Numer. Simul. in Cont. Mech., Lecture Notes in Comput. Sci. Engin., Springer Verlag, ed. by T. Miyoshi, pp.1-10 (2001).

[11] Yamamoto, N., A numerical verification method for solutions of boundary value problems with local uniqueness by Banach's fixed point theorem, SIAM J. Numer. Anal. 35 (1998), 2004-2013.