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Dynamic Pattern Recognition through Bio-inspired Oscillatory CNNs

Marco Gilli, Michele Bonnin, and Fernando Corinto

Department of Electronics
Technical University of Torino
Torino, Italy
Email: marco.gilli@polito.it, michele.bonnin@polito.it, fernando.corinto@polito.it

Abstract—Many studies in neuroscience have shown that nonlinear dynamic networks represent a bio-inspired models for information and image processing. Recent studies on the thalamo-cortical system have shown that weakly connected oscillatory networks have the capability of modelling the architecture of a neurocomputer. In particular they have associative properties and can be exploited for dynamic pattern recognition. In this manuscript the global dynamic behavior of weakly connected cellular networks of oscillators is investigated. It is assumed that each cell admits of a Lur'e description. In case of weak coupling the main dynamic features of the network are revealed by the phase deviation equation (i.e. the equation that describes the phase deviation due to the weak coupling). Firstly a very accurate analytic expression of the phase deviation equation is derived, via the joint application of the describing function technique and of Malkin's Theorem. Then it is shown that the total number of periodic limit cycles, with their stability properties, can be estimated through the analysis of the phase deviation equation.

1. Introduction

Weakly connected oscillatory networks represent bio-inspired architectures for information and image processing. Recent studies in neuroscience have shown that some significant features of the visual systems, like the binding problem [1], can be investigated, by exploiting nonlinear dynamic network models [2]. Some studies on the thalamo-cortical system have suggested new architectures for neurocomputers, that consist of locally coupled arrays of oscillators, with a periodic and/or complex dynamic behavior (including the possibility of chaos) [3, 4]. In particular, it has been shown that nonlinear oscillatory networks exhibit associative properties and can be exploited for dynamic pattern recognition [3, 4].

In many cases such networks can be adequately modelled as cellular neural/nonlinear networks (CNNs), a new paradigm of analog dynamic processors, that was introduced some years ago in the electrical engineering community [5, 6]. CNNs are described as 2 or n-dimensional arrays of mainly identical nonlinear dynamical systems (called cells), that are locally interconnected. In most applications the connections are specified through space-invariant templates (that consist of small sets of parameters identical for all the cells). The local connectivity has allowed the realization of several high-speed VLSI chips [7].

The mathematical model of a CNN consists of a large system of locally coupled nonlinear ordinary differential equations (ODEs), that may exhibit a rich spatio-temporal dynamics, including several attractors and bifurcation phenomena [8]. For this reason CNN dynamics has been mainly investigated through time-domain numerical simulation. Recently some spectral techniques have been applied to space-invariant CNNs, in order to characterize some space-time phenomena (see [8] and in particular [9, 10]). However the proposed methods are not suitable for characterizing the global dynamic behavior of complex networks, that exhibit a large number of attractors.

Weakly connected oscillatory networks can be investigated through the phase deviation equation, [4] i.e. the equation that describes the evolution of the phase deviations, due to the weak coupling. We have employed this method for investigating one-dimensional weakly connected networks, composed by third order oscillators (Chua's circuits) [11]. In particular we have shown that an accurate analytic expression of the phase deviation equation can be derived, via the joint application of the describing function technique and of Malkin's Theorem.

In this manuscript we focus on oscillatory patterns in weakly connected networks, employed as dynamic associative memories [3]. We firstly derive an accurate analytic expression of the phase deviation equation for generic weakly connected networks, composed by nonlinear oscillators, that admit of a Lur'e type description [12]. Then we show that a detailed analysis of the phase deviation equation allows one to accurately estimate the total number of periodic limit cycles with their stability properties.

2. Weakly Connected Networks

A weakly connected network (WCN), [4] composed by n cells of dynamical order m, is described by the following system of nonlinear ordinary differential equations (ODEs) ($1 \leq i \leq n$):

$$\dot{X}_i = F_i(X_i) + \varepsilon\, G_i(\boldsymbol{X}), \qquad \boldsymbol{X} = [X_1^T, \dots X_n^T]^T \quad (1)$$

where X_i represents the state vector of each cell, $F_i : \boldsymbol{R}^m \to \boldsymbol{R}^m$, $G_i : \boldsymbol{R}^{m\times n} \to \boldsymbol{R}^m$, T denotes transposition and ε represents a small parameter that guarantees a weak connection among the cells.

We assume that each uncoupled cell admits of a Lur'e representation [12], and in particular that its state equations can be recast as follows:

$$\begin{aligned} \dot{x}_i^a &= A_{11}^i\, x_i^a + \boldsymbol{A}_{12}^i\, X_i^b + f_i(x_i^a) \\ \dot{X}_i^b &= \boldsymbol{A}_{21}^i\, x_i^a + \boldsymbol{A}_{22}^i\, X_i^b \end{aligned} \quad (2)$$

where $x_i^a \in \boldsymbol{R}$ is a scalar component of X_i, $X_i^b \in \boldsymbol{R}^{m-1}$ represents the collection of the other components of X_i, $A_{11}^i \in \boldsymbol{R}$, $\boldsymbol{A}_{12}^i \in \boldsymbol{R}^{1,m-1}$, $\boldsymbol{A}_{21}^i \in \boldsymbol{R}^{m-1,1}$, $\boldsymbol{A}_{22}^i \in \boldsymbol{R}^{m-1,m-1}$ and $f_i(\cdot)$ is a scalar Lipschitz nonlinear function.

By exploiting (2) a linear relationship between $X_i^b(t)$ and $x_i^a(t)$ is readily derived:

$$X_i^b(t) = (D - \boldsymbol{A}_{22}^i)^{-1}\, \boldsymbol{A}_{21}^i\; x_i^a(t) \quad (3)$$

where D denotes the first order time-differential operator.

This allows one to rewrite equations (2) in term of the sole scalar variable x_i^a:

$$L_i(D)\, x_i^a(t) = f_i[x_i^a(t)] \quad (4)$$

where $L_i(D)$ has the following expression:

$$L_i(D) = D - A_{11}^i - \boldsymbol{A}_{12}^i\, (D - \boldsymbol{A}_{22}^i)^{-1}\, \boldsymbol{A}_{21}^i \quad (5)$$

We assume that each cell is only coupled to the cells belonging to its neighborhood of radius r and that the coupling is space-invariant and only involves the scalar variables x_i^a. By denoting with C_k the corresponding space-invariant template, the resulting WCN is described by the following simplified system of Lur'e like equations:

$$L^i(D)\, x_i^a(t) = f_i[x_i^a(t)] + \varepsilon \sum_{k=-r}^{k=r} C_k\, x_{i+k}^a(t) \quad (6)$$

It turns out that only one component of $\boldsymbol{G}_i$ is different from zero, i.e.

$$G_i(\boldsymbol{X}) = \begin{pmatrix} \sum_{k=-r}^{k=r} C_k\, x_{i+k}^a(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (7)$$

We assume that, in absence of coupling, each cell only exhibits the following invariant limit sets: a finite number of unstable equilibrium points, a finite number of either stable or unstable periodic limit cycles and at least one asymptotically stable limit cycle.

According to the these assumptions, we focus on a set of parameters and initial conditions such that the trajectory of each uncoupled cell is a periodic (either stable or unstable) limit cycle, described by a regular curve $\gamma_i(t) \subset R^m$. If we denote by ω_i and $\theta_i \in S^1 = [0,\ 2\pi[$ the angular frequency and the phase respectively of each limit cycle $\gamma_i(t)$, then the WCN admits of the following description in term of phase variables:

$$\theta_i(t) = \omega_i\, t + \phi_i(t) \quad (8)$$

where $\phi_i(t) \in S^1$ represents the phase deviation from the natural oscillations, due to weak coupling.

If the angular frequencies ω_i are commensurable, then Malkin's Theorem [4] provides an explicit way for deriving the system of differential equations, that governs the phase deviation evolution. For the sake of the completeness and for introducing the proper notations, we report here a simplified version of Malkin's Theorem [4] (see also [11]).

Theorem 1 (Malkin's Theorem for weakly coupled oscillator, with commensurable angular frequencies): Consider a WCN described by (1) and assume that each uncoupled cell

$$\dot{X}_i = F_i(X_i), \quad X_i \in \boldsymbol{R}^m, \qquad (1 \le i \le n) \quad (9)$$

has a hyperbolic (either stable or unstable) periodic orbit $\gamma_i(t) \subset R^m$ of period T_i and angular frequency $\omega_i = 2\pi/T_i$. Let $\tau = \varepsilon\, t$ be slow time and let $\phi_i(\tau)$ be the phase deviation from the natural oscillation $\gamma_i(t),\ t \ge 0$. Then the vector of phase deviation $\phi = (\phi_1,\ \phi_2,\ ...,\ \phi_n)^T$ is a solution to:

$$\phi_i' = H_i(\phi - \phi_i,\ \varepsilon) \qquad (1 \le i \le n) \quad (10)$$

where $\phi - \phi_i = (\phi_1 - \phi_i,\ ...,\ \phi_n - \phi_i)^T \in [0,\ 2\pi[^n = \boldsymbol{T}^n$, $' = \frac{d}{d\tau}$ and

$$\begin{aligned} H_i(\phi - \phi_i, 0) &= \frac{1}{T} \int_0^T Q_i^T(t)\, G_i \left[\gamma \left(t + \frac{\phi - \phi_i}{\omega} \right) \right] dt \\ \gamma \left(t + \frac{\phi - \phi_i}{\omega} \right) &= \left[\gamma_1^T \left(t + \frac{\phi_1 - \phi_i}{\omega_1} \right),\ ...,\gamma_n^T \left(t + \frac{\phi_n - \phi_i}{\omega_n} \right) \right]^T \end{aligned} \quad (11)$$

being T the minimum common multiple of $T_1,\ T_2,\ ...,\ T_n$. In the above expression (11) $Q_i(t) \in \boldsymbol{R}^m$ is the unique nontrivial T_i-periodic solution to the linear time-variant system:

$$\dot{Q}_i(t) = -[DF_i(\gamma_i(t))]^T\, Q_i(t) \quad (12)$$

$$Q_i^T(0) F_i(\gamma_i(0)) = 1 \quad (13)$$

In order to apply Malkin's Theorem to weakly connected networks and to compute the phase deviation equation (10) the knowledge of $\gamma_i(t)$ (i.e. the limit cycle trajectories in absence of coupling) is required. Our method is based on the idea that $\gamma_i(t)$ can be approximately computed by exploiting the describing function technique (the conditions under which this assumption is reasonable are given in [12] and [13]). More precisely the proposed methods consists of three fundamental steps, that are described in the following three subsections.

2.1. Describing function approximation of $\gamma_i(t)$

The scalar variable $x_i^a(t)$ of the Lur'e model (4) is represented through a bias term and a single harmonic, with suitable amplitude and angular frequency:

$$x_i^a(t) \approx \hat{x}_i^a(t) = A_i + B_i \sin(\omega_i\, t) \tag{14}$$

where A_i denotes the bias, B_i the amplitude of the first harmonic, and ω_i is the angular frequency.

The output of the nonlinear function $f(\cdot)$, when the input is (14), admits of the following first harmonic representation (that in several cases can be expressed through a close analytical form):

$$f_i[\hat{x}_i^a(t)] \;\approx\; F^A(A_i, B_i) \;+\; F^B(A_i, B_i)\, \sin(\omega_i\, t) \tag{15}$$

where:

$$\begin{aligned} F_i^A(A_i, B_i) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f_i\, [A_i + B_i \sin(\theta)]\, d\theta \\ F_i^B(A_i, B_i) &= \frac{1}{\pi} \int_{-\pi}^{\pi} f_i\, [A_i + B_i \sin(\theta)]\; \sin(\theta)\; d\theta \end{aligned} \tag{16}$$

The parameters A_i, B_i and ω_i are the solution of the describing function system shown below:

$$L_i(0)\, A_i = F_i^A(A_i, B_i) \tag{17}$$

$$\boldsymbol{Re}[L_i(j\omega_i)]\; B_i = F_i^B(A_i, B_i) \tag{18}$$

$$\boldsymbol{Im}[L_i(j\omega_i)] = 0 \tag{19}$$

Once $\hat{x}_i^a(t)$ is known, the first harmonic approximation of $\gamma_i(t)$, i.e. $\hat{\gamma}_i(t) = [\hat{x}_i^a(t), [\hat{X}_i^b(t)]^T]^T$, is determined by deriving $\hat{X}_i^b(t)$, via the linear differential relation (3).

2.2. Describing function approximation of $Q_i(t)$

According to Malkin's Theorem, $Q_i = [q_i^a, [Q_i^b]^T]^T$ is the unique T_i-periodic solution to the linear time-variant system (12), satisfying the normalization condition (13):

$$\begin{pmatrix} \dot{q}_i^a(t) \\ \dot{Q}_i^b(t) \end{pmatrix} = -\begin{pmatrix} A_{11}^i + f_i'[x_i^a(t)] & (\boldsymbol{A}_{21}^i)^T \\ (\boldsymbol{A}_{12}^i)^T & (\boldsymbol{A}_{22}^i)^T \end{pmatrix} \begin{pmatrix} q_i^a(t) \\ Q_i^b(t) \end{pmatrix} \tag{20}$$

Hence a linear relationship holds between $Q_i^b(t)$ and q_i^a:

$$Q_i^b(t) = -[D + (\boldsymbol{A}_{22}^i)^T]^{-1}\, (\boldsymbol{A}_{12}^i)^T\; q_i^a(t) \tag{21}$$

By substituting (21) in (20) and by exploiting (5), it is easily proved that the following scalar Lur'e like variational equation holds for $q_i^a(t)$:

$$L_i(-D)\; q_i^a(t) = \; f'[x_i^a(t)]\; q_i^a(t) \tag{22}$$

The aim of this Section is to find a first harmonic approximation $\hat{q}_i^a(t)$ of $q_i^a(t)$. The main result is enunciated in the following Proposition. The proof, that for lack of space is not reported here, can be found in [14].

Proposition: If the limit cycle $\gamma_i(t)$ is hyperbolic and all the harmonics of order higher than one are neglected, then the following expression holds for $\hat{q}_i^a(t)$:

$$\hat{q}_i^a(t) = \delta_i(\omega_i)\; \cos(\omega_i t) \tag{23}$$

where

$$\delta_i(\omega_i) = \frac{1}{\omega_i\, B_i\, \{\boldsymbol{Im}[\boldsymbol{N}_i^T(j\omega_i)] \cdot \boldsymbol{Im}[\boldsymbol{M}_i(j\omega_i)]\}} \tag{24}$$

$$\boldsymbol{M}_i(j\omega_i) = (j\omega_i - \boldsymbol{A}_{22}^i)^{-1}\, \boldsymbol{A}_{21}^i \tag{25}$$

$$\boldsymbol{N}_i(j\omega_i) = -[j\omega_i + (\boldsymbol{A}_{22}^i)^T]^{-1}\, (\boldsymbol{A}_{12}^i)^T \tag{26}$$

2.3. Phase deviation equation

We show that an explicit and very accurate expression of the phase deviation equation can be derived by substituting in (11) the describing function approximation of $\gamma_i(t)$ and $Q_i(t)$, provided in the previous subsections. By remembering that, according to (7), only one component of $G_i(\boldsymbol{X})$ is different from zero, we obtain:

$$\begin{aligned} \phi_i' &\approx H_i(\phi - \phi_i, 0) = \frac{1}{T} \int_0^T Q_i^T(t)\, G_i \left[\gamma\left(t + \frac{\phi - \phi_i}{\omega}\right)\right] dt \\ &\approx \frac{1}{T} \int_0^T \hat{q}_i^a(t) \sum_{k=-r}^{k=r} C_k\, \hat{x}_{i+k}^a \left(t + \frac{\phi_{i+k} - \phi_i}{\omega_{i+k}}\right) dt \\ &= \frac{1}{T} \int_0^T \delta_i \cos(\omega_i t) \sum_{k=-r}^{k=r} C_k\, [A_{i+k} + B_{i+k} \sin(\omega_{i+k} t + \phi_{i+k} - \phi_i)]\, dt \\ &= \frac{\delta_i}{2} \sum_{k\in[-r,r],\, \omega_i=\omega_{i+k}} C_k\, B_{i+k}\; \sin[\phi_{i+k} - \phi_i] \\ &= \frac{V_i(\omega_i)}{B_i} \sum_{k\in[-r,r],\, \omega_i=\omega_{i+k}} C_k\, B_{i+k}\; \sin[\phi_{i+k} - \phi_i] \end{aligned} \tag{27}$$

where, according to (24) the following expression holds for $V_i(\omega_i)$:

$$V(\omega_i) = \frac{1}{2\, \omega_i\, \{\boldsymbol{Im}[\boldsymbol{N}_i^T(j\omega_i)] \cdot \boldsymbol{Im}[\boldsymbol{M}_i(j\omega_i)]\}} \tag{28}$$

It is worth noting that the phase of the i-th oscillator is only influenced by the oscillators with identic angular frequency: this is in agreement with the well known and general results reported in [4].

If all the cells are identical (i.e. $\forall\, k\, :\; B_{i+k} = B_i,\;\; \omega_i = \omega_{i+k}$), the equation above admits of the following simplified form:

$$\phi_i' \approx V(\omega) \sum_{k=-r}^{k=r} C_k\; \sin(\phi_{i+k} - \phi_i) \tag{29}$$

where

$$V(\omega) = \frac{1}{2\,\omega\,\{\boldsymbol{Im}[\boldsymbol{N}^T(j\omega)]\cdot\boldsymbol{Im}[\boldsymbol{M}(j\omega)]\}} \qquad (30)$$

Remark 1: Equation (27) reduces a rather complex network of oscillators to a simple Kuramoto-like model [15], that can be analytically dealt with. This opens the possibility of developing new applications, that exploit the rich dynamic behavior of nonlinear dynamic arrays, including dynamic pattern recognitions and dynamic associative memories [3].

Remark 2: Each limit cycle (either stable or unstable) of the weakly connected network corresponds to an equilibrium point of the phase deviation equation (27) (see [4]). The analytical Kuramoto-like form of (27) allows one to detect the total number of equilibrium points and hence to estimate the total number of periodic limit cycles.

Remark 3: The stability analysis of each limit cycle arises from the stability analysis of each equilibrium point of (27), by computing the corresponding Jacobian matrix. The latter admits of a simple analytical form and hence is particularly suitable for stability analysis.

Remark 4: The case under study is a one dimensional array of oscillators, with local space-invariant connections. However the proposed approach allows one to explicitly derive a very accurate analytical approximation of the phase deviation equation for more complex networks. In particular the general case of two-dimensional fully connected arrays, composed by oscillators with different and commensurable angular frequencies can be dealt with.

3. Conclusions

Weakly connected oscillatory networks are bio-inspired models for information processing. In particular they have associative properties and can be exploited for dynamic pattern recognition. We have considered space-invariant weakly connected networks, composed by nonlinear oscillators that admit of a Lur'e type description. We have shown that an accurate analytical expression of the phase deviation equation (i.e. the equation that describes the phase evolution, due to the weak coupling) can be derived via the joint application of the describing function technique and of Malkin's Theorem. This extends the results that we presented in a previous work, which only apply to one-dimensional array of third-order oscillators (in particular Chua's circuits) [11]. Then we have shown that a detailed analytical study of the phase deviation equation allows one to accurately estimate the total number of limit cycles and their stability characteristics.

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