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Adomian decomposition method as a tool for numerical studying multi-scroll hyperchaotic attractors

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Abstract – This paper deals with the numerical study of multi-scroll hyperchaotic attractors via the Adomian decomposition method. This objective is achieved by exploiting Hermite interpolating polynomials as nonlinearities in coupled Chua's circuits. Several examples of generation of multi-scroll attractors highlight the capabilities of the proposed technique.

1. Introduction

Recently, Chua's circuits with modified characteristics for the nonlinear resistors have been introduced, with the aim of generating more complex chaotic attractors [1]-[3]. In particular, in [2] novel examples of chaotic attractors with n-double scroll have been obtained by exploiting a sine-type function instead of the PWL characteristic of the Chua diode. Moreover, in [4]-[5] the dynamic properties and the generation of new hyperchaotic multi-scroll attractors have been illustrated. More precisely, by considering coupled Chua's circuits with sine-type characteristics, it is shown that the number of scrolls can be easily designed by modifying suitable parameters related to the nonlinearities.

In this paper a further contribution to the topic of multi-scroll attractors is given. The starting point of the proposed approach is the recently introduced Adomian technique [6]. This method is a decomposition technique that doesn't change the real physical problem into a convenient one for use with linear theory, but provides series solutions which generally converge very rapidly. In particular, the Adomian method provides immediate and visible symbolic terms of analytic solutions, as well as numerical solutions to nonlinear differential equations without linearization or discretization (as in the Runge-Kutta method).

Since complex attractors are described by strongly nonlinear differential equations, the aim of this paper is to study hyperchaotic multi-scroll dynamics by utilizing the Adomian decomposition method. In particular, this paper analyses 2D-scroll hyperchaotic attractors generated by two coupled Chua's circuits characterized by a new type of nonlinearities. Such nonlinearities belong to the class of the Hermite interpolating polynomials [7] and are well suited for calculating the Adomian polynomials required for obtaining the series solution of the Adomian approach. Numerical results demonstrate that the method is quite accurate and readily implemented in studying complex multi-scrolls dynamics.

The paper is organized as follows. In Sec.2 the state equations of the proposed coupled Chua's circuits are reported and the Adomian method is introduced as a tool for numerical studying multi-scroll hyperchaotic attractors. In Sec.3, Hermite polynomials are exploited as new type of nonlinearities and it is shown how hyperchaotic 3x4-scroll attractors can be generated by combining the chaotic 3-scroll attractor (related to the first circuit) with the chaotic 4-scroll attractor (related to the second circuit). By generalizing such results, in Sec.4 hyperchaotic nxm-scroll attractor are generated by combining the chaotic n-scroll attractor related to the first circuit with the chaotic m-scroll attractor related to second circuit.

2. A Tool for Studying Multi-Scroll Attractors

2.1 Equations of Coupled Chua's Circuits

The proposed hyperchaotic circuit is the 6[th] order system obtained by coupling two Chua's circuits and by replacing the two Chua diodes with two polynomial nonlinearities. The state equations can be written in dimensionless form as:

$$\begin{aligned}
\dot{x}_1 &= \alpha\left[x_2 - H_1(x_1)\right]\\
\dot{x}_2 &= x_1 - x_2 + x_3 + M(x_5 - x_2)\\
\dot{x}_3 &= -\beta x_2\\
\dot{x}_4 &= \alpha\left[x_5 - H_2(x_4)\right]\\
\dot{x}_5 &= x_4 - x_5 + x_6 + M(x_2 - x_5)\\
\dot{x}_6 &= -\beta x_5
\end{aligned} \tag{1}$$

where α = 10.814, β = 14 and M=0.25, whereas the smooth nonlinear functions $H_1(\cdot)$ and $H_2(\cdot)$ are suitable Hermite polynomials that play the same role of the sine-type nonlinearities illustrated in [4]. Details about the Hermite polynomials will be given in Sec.3.

2.2 Adomian Decomposition Method

Consider the equation $F(\boldsymbol{u}(t)) = \boldsymbol{g}(t)$, where F represents a nonlinear ordinary differential operator involving both linear and nonlinear terms, whereas $\boldsymbol{u}(t) = \left(u^1(t), u^2(t), ..., u^m(t)\right)^T$ and $\boldsymbol{g}(t) = \left(g_1(t), g_2(t), ..., g_m(t)\right)^T$ are real vector functions. The decomposition method requires that the nonlinear operator F be separated into three terms $F = D + L + N$, where N is a nonlinear vector and ($D + L$) form the linear vector. Here D is chosen to be easily invertible and L is the remainder of the linear term. For convenience, D may be taken as the highest order derivative. Thus, the initial value problem may be written $D\boldsymbol{u} + L\boldsymbol{u} + N\boldsymbol{u} = \boldsymbol{g}(t)$ with $\boldsymbol{u}(t_0) = \boldsymbol{u}_0$. The Adomian method is based on applying the inverse operator D^{-1} formally to the expression $D\boldsymbol{u} = \boldsymbol{g}(t) - L\boldsymbol{u} - N\boldsymbol{u}$ [6]. It should be noted that the linear operator D may have a non-trivial kernel and D^{-1} is then not unique. For example if D is a second derivative operator, a typical kernel element k might take the form $k = A + Bt$. Therefore, the application of the inverse operator D^{-1} gives $\boldsymbol{u} = \boldsymbol{k} + D^{-1}\boldsymbol{g}(t) - D^{-1}L\boldsymbol{u} - D^{-1}N\boldsymbol{u}$ [6]. Now, the vector solution $\boldsymbol{u}(t)$ and the $N\boldsymbol{u}$ term can be written in the decomposition form as:

$$\boldsymbol{u} = \sum_{i=0}^{\infty} \boldsymbol{u}_i = \sum_{i=0}^{\infty} \begin{pmatrix} u_i^1 \\ u_i^2 \\ \cdots \\ u_i^m \end{pmatrix} \tag{2}$$

$$N\boldsymbol{u} = \sum_{i=0}^{\infty} \boldsymbol{A}_i = \sum_{i=0}^{\infty} \begin{pmatrix} A_i^1 \\ A_i^2 \\ \cdots \\ A_i^m \end{pmatrix} \tag{3}$$

where the vector components of $\boldsymbol{A}_i$ are the Adomian polynomials [6]. Therefore, the components of the solution (2) can be defined recursively as follows:

$$\begin{aligned} &\boldsymbol{u}_0 = \boldsymbol{k} + D^{-1}\boldsymbol{g}(t) \\ &\boldsymbol{u}_1 = -D^{-1}L\boldsymbol{u}_0 - D^{-1}\boldsymbol{A}_0 \\ &\boldsymbol{u}_2 = -D^{-1}L\boldsymbol{u}_1 - D^{-1}\boldsymbol{A}_1 \qquad i \geq 1. \\ &\cdots \\ &\boldsymbol{u}_i = -D^{-1}L\boldsymbol{u}_{i-1} - D^{-1}\boldsymbol{A}_{i-1} \\ &\cdots \end{aligned} \tag{4}$$

In order to determine the expression of the Adomian polynomials $\boldsymbol{A}_i$, in this paper the formula reported in [8] is utilized:

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(v_n(\lambda)\right)\right]_{\lambda=0}, n = 0,1,... \tag{5}$$

where $v_n(\lambda) = \sum_{i=0}^{n} \lambda^i u_i$. Note that (5) can be well adapted to calculate $\boldsymbol{A}_i$ by applying symbolic computation. Now, by applying the Adomian decomposition method to the considered coupled Chua's circuits (1), the infinite series solution of (1) can be written as:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \end{pmatrix} = \left(\sum_{i=0}^{\infty} x_{1,i} \ \ \sum_{i=0}^{\infty} x_{2,i} \ \ \sum_{i=0}^{\infty} x_{3,i} \ \ \sum_{i=0}^{\infty} x_{4,i} \ \ \sum_{i=0}^{\infty} x_{5,i} \ \ \sum_{i=0}^{\infty} x_{6,i} \right)^T =$$

$$= \begin{pmatrix} x_1(t_0) \\ x_2(t_0) \\ x_3(t_0) \\ x_4(t_0) \\ x_5(t_0) \\ x_6(t_0) \end{pmatrix} + \int_{t_0}^{t} \begin{pmatrix} \alpha x_2 \\ x_1 - x_2(1+M) + x_3 + Mx_5 \\ -\beta x_2 \\ \alpha x_5 \\ Mx_2 + x_4 - x_5(1+M) + x_6 \\ -\beta x_5 \end{pmatrix} dt - \alpha \int_{t_0}^{t} \begin{pmatrix} H_1(x_1) \\ 0 \\ 0 \\ H_2(x_4) \\ 0 \\ 0 \end{pmatrix} dt \tag{6}$$

where the Hermite polynomials $H_1(x_1)$ and $H_2(x_4)$ are expanded by using the formula (5). Finally, the six-terms series of the infinite series (6) can be obtained:

$$\begin{aligned} \tilde{x}_j(t) = \sum_{i=0}^{5} x_{j,i} &= k_{j,0} + k_{j,1}(t-t_0) + k_{j,2}\frac{(t-t_0)^2}{2!} + \\ &+ k_{j,3}\frac{(t-t_0)^3}{3!} + k_{j,4}\frac{(t-t_0)^4}{4!} + k_{j,5}\frac{(t-t_0)^5}{5!}, \ j = 1,...,6 \end{aligned} \tag{7}$$

where the expressions of the parameters $k_{j,i}$ depends on the parameters and the initial conditions of the circuit (1).

Differently from Runge-Kutta numerical method, the solution (7) has been obtained without linear approximation but preserving the circuit nonlinearity. In particular, note that:

1. even though the solution (7) constitutes a truncation of the infinite series solution (6) up to a few terms, this is enough to give an accurate result, as shown in Sec.3;
2. for large values of time t, it can still be used the six-term solution (7) by dividing the [t_0, t] interval into subintervals [t_k, t_{k+1}] and then use the same formula (7) but taking $t_0 = t_k$ and $\tilde{x}_j(t_0) = \tilde{x}_j(t_k)$ $(j = 1,\ldots,6)$ in order to evaluate $\tilde{x}_j(t_{k+1})$ $(j = 1,\ldots,6)$;
3. the Adomian polynomials in (6) and the constants in (7) can be calculated by using symbolic tools, usually available in numerical packages like Matlab, Mathematica or Maple.

3. Generation of 3x4-Scroll Attractors

In order to show how the Adomian method works, the attention is focused on the generation of hyperchaotic 3x4-scroll attractors. The results reported in [4] show that such 3x4-scroll attractor can be generated by using a combination of a sine-type nonlinearity and a piece-wise linear function. Since this type of nonlinearity cannot be expanded by using the Adomian method, herein the nonlinearities adopted in [4] are replaced by Hermite interpolating polynomials. In particular, in order to generate the 3x4-scroll attractor, the smooth Hermite polynomials $H_1(\cdot)$ (related to the 3-scroll attractor of the first Chua's circuit) and $H_2(\cdot)$ (related to the 4-scroll attractor of the second Chua's circuit) have to be chosen as:

$$H_1(x_1) = a_1x_1^{13} + b_1x_1^{11} + c_1x_1^9 + d_1x_1^7 + e_1x_1^5 + f_1x_1^3 + g_1x_1 \tag{8a}$$

$$\begin{aligned} H_2(x_4) = &a_2x_4^{17} + b_2x_4^{15} + c_2x_4^{13} + d_2x_4^{11} + e_2x_4^9 + f_2x_4^7 + \\ &+ g_2x_4^5 + h_2x_4^3 + i_2x_4 \end{aligned} \tag{8b}$$

where

a_1 = 2.8404090e-11;	a_2 = -1.615715e-15;	
b_1 = -1.00365277e-8;	b_2 = 7.657509e-13;	
c_1 = 1.25449979e-6;	c_2 = -1.62497141e-10;	
d_1 = -7.5336678e-5;	d_2 = 2.0340009e-8;	(8c)
e_1 = 2.3129468e-3;	e_2 = -1.62369019e-6;	
f_1 = -3.222156e-2;	f_2 = 8.1554746e-5;	
g_1 = 1.329135e-1;	g_2 = -2.357797e-3;	
	h_2 = 3.233495e-2;	
	i_2 = -1.32914e-1;	

Note that parameters (8c) of the Hermite polynomials (8a)-(8b) have been calculated in order to osculate the sine-type nonlinearity reported in [4] (see Fig.1 in the case of 4-scroll). This choice enables to obtain smooth nonlinearities and, at the same time, to avoid the piecewise characteristic. Additionally, the chosen Hermite polynomials have the same equilibrium points of the sine-type nonlinearities in [4]. Finally, note that the equilibria of the Hermite polynomials share the same dynamic properties of the equilibria of the sine-type nonlinearities [4]. Now, the series solution (7) of system (1) with nonlinearities (8) (not reported herein for lack of space) is readily obtained by using the symbolic tool available in the Matlab package.

By exploiting the results illustrated in [4], it can be shown that the circuit (1) with Hermite polynomials (8) possesses 35 equilibrium points in the form:

$$x_{eq} = (x_{1eq}, 0, -x_{1eq}, x_{4eq}, 0, -x_{4eq}) \in \Re^6 \qquad (9)$$

where $(x_{1eq}, 0, -x_{1eq})$ and $(x_{4eq}, 0, -x_{4eq})$ are the equilibria of the first circuit and of the second circuit, respectively. Herein, as in [4], it can be proved that the stability properties of all the 35 equilibria (9) can be derived from the stability of the equilibria of the single Chua's circuit. In particular, the hyperchaotic 3x4-scroll attractor (Fig.2(a)) can be generated by combining the chaotic 3-scroll attractor of the first circuit (Fig.2(b)) with the chaotic 4-scroll attractor of the second circuit (Fig.2(c)).

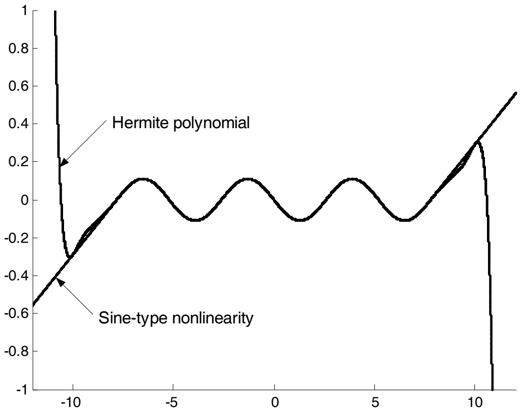
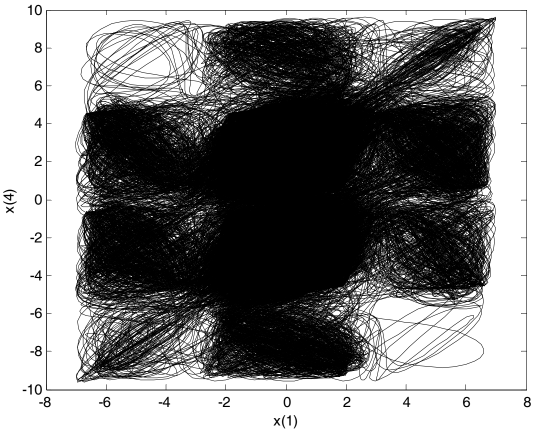
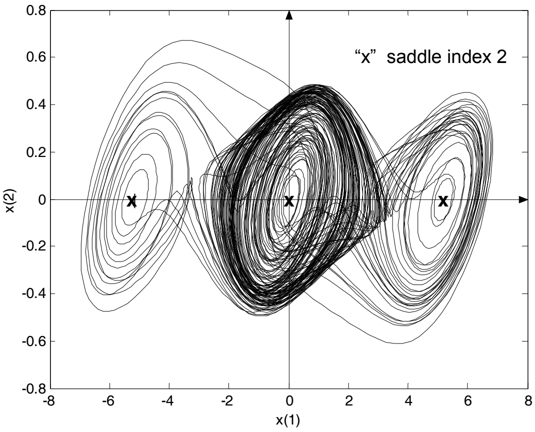


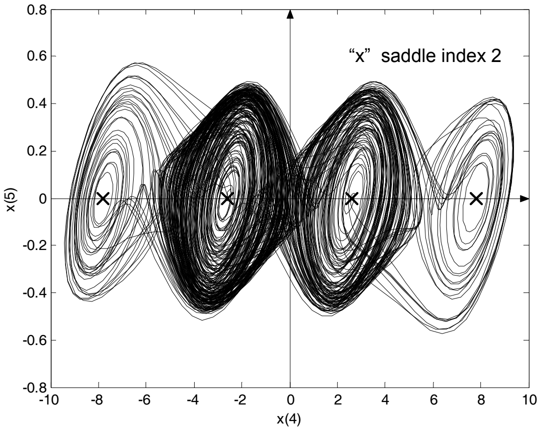
Figure 1. Generation of 4-scroll attractor in the second Chua's circuit: the Hermite polynomial $H_2(\cdot)$ osculates the sine-type nonlinearity.



(a)



(b)



(c)

Figure 2. Dynamics of coupled Chua's circuits; (a): hyperchaotic 3x4-scroll attractor on (x_1, x_4)-plane; (b): chaotic 3-scroll attractor on (x_1, x_2)-plane; (c): chaotic 4-scroll attractor on (x_4, x_5)-plane.

As in [4], by taking into account that *saddle points index r* are characterized by *r* eigenvalues with positive real parts, some properties about the stability-type of each equilibrium point can be stated. Namely:

Property 1: The combination of:
saddle point index 1 (x_{1eq}) of the 1st single Chua's circuit;
saddle point index 1(x_{4eq}) of the 2nd single Chua's circuit;
generates in coupled Chua's circuits

saddle point index 2 (x_{eq})

characterized by 2 positive real eigenvalues.

Property 2: The combination of:
saddle point index 1 (x_{1eq}) of the 1st single Chua's circuit;
saddle point index 2(x_{4eq}) of the 2nd single Chua's circuit;
generates in coupled Chua's circuits

saddle point index 3 (x_{eq})

characterized by 2 complex and 1 real eigenvalues with positive real parts. The same property holds if the stability-type of x_{1eq} and x_{4eq} is exchanged.

Property 3: The combination of:
saddle point index 2 (x_{1eq}) of the 1st single Chua's circuit;
saddle point index 2(x_{4eq}) of the 2nd single Chua's circuit;
generates in coupled Chua's circuits

saddle point index 4 (x_{eq})

characterized by 4 complex eigenvalues with 4 positive real parts.

Previous properties, which state the stability of all the 35 equilibria of coupled Chua's circuits starting from the stability of the equilibria of the single circuit, make perceive the possibility that hyperchaotic 3x4-scroll attractor can be generated by combining chaotic 3-scroll attractor (first circuit) with chaotic 4-scroll attractor (second circuit). To this purpose, the following property can be stated.

Property 4: Similarly to single Chua's circuit, where 3-scroll and 4-scroll attractors are generated only around equilibria having all the complex eigenvalues with positive real parts (*saddle points index 2*), in the case of coupled Chua's circuits (1) with Hermite polynomials (8) the scrolls of the 3x4-scroll attractor are generated only around the 12 equilibria having all the complex eigenvalues with positive real parts (*saddle points index 4*).

Details about the proof of Property 4 can be found in [4].

4. Generation of Multi-Scroll Attractors

By generalizing previous results, it can be readily shown that, given two coupled Chua's circuits with Hermite nonlinearities, it is possible to choose the numbers n and m of the scrolls of the first and second circuit, respectively, so that hyperchaotic nxm-scroll attractors are obtained. For example, the hyperchaotic 8x4-scroll attractor is reported in Fig.3.

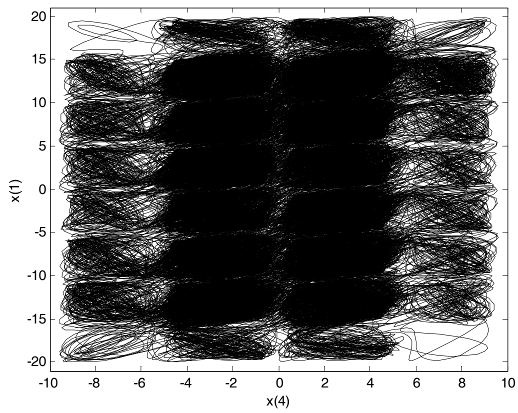


Figure 3. Generation of the hyperchaotic 8x4-scroll attractor.

Similarly, the hyperchaotic 9x9-scroll attractor is reported in Fig.4. In particular, Figs.4(*a*)-(b) highlight how the dynamics of the 81 scrolls are generated starting from the origin, whereas Fig.4(*c*) shows the "final" attractor.

5. Conclusions

This paper has focused on the numerical study of multi-scroll hyperchaotic attractors by applying the Adomian decomposition method. This objective has been achieved by taking Hermite interpolating polynomials as nonlinearities in coupled Chua's circuits. Design examples have been reported to illustrate how different multi-scroll attractors can be systematically generated.

6. References

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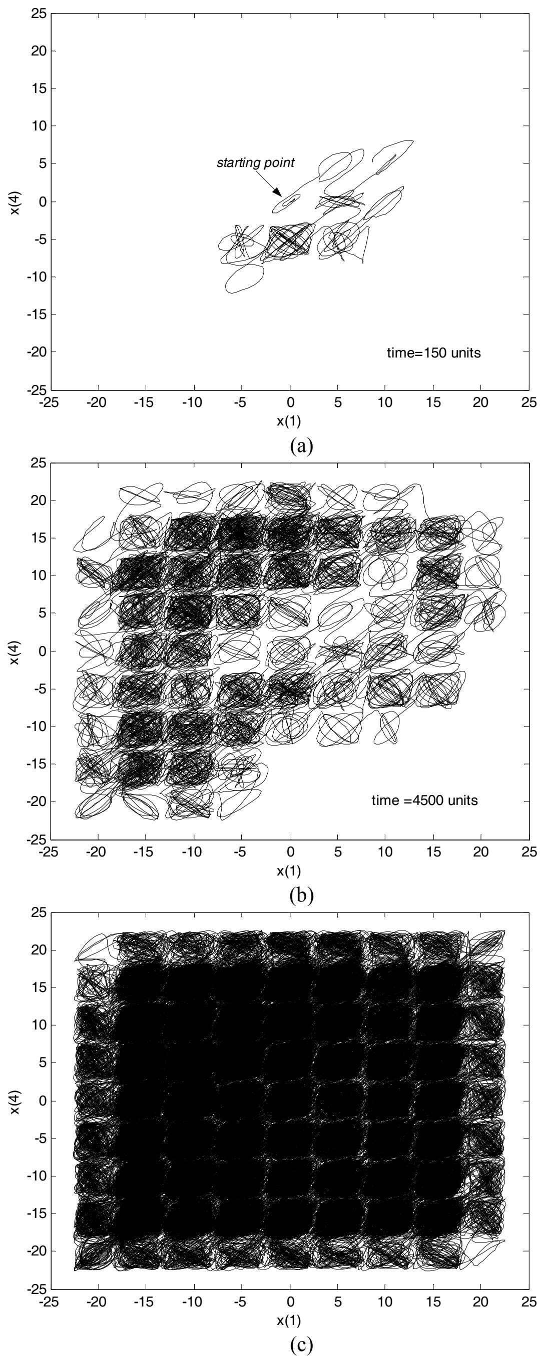

Figure 4. Hyperchaotic 9x9-scroll attractor on (x_1, x_4)-plane. (*a*): time t=150; (*b*): time t=4500; (*c*): final attractor.