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Benefits of the Input Minimum Phase Property for Linearization of Nonlinear Systems

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Abstract—Linear approximations of nonlinear systems can be obtained by fitting a linear model to data from a nonlinear system, for example, using the prediction-error method. In many situations, the type of linear model and the model orders are selected after estimating several models and evaluating them using various validation techniques. Two commonly used validation methods for linear models are spectral and residual analysis. Unfortunately, these methods will not always work if the true system is nonlinear. However, if the input can be viewed as if it has been generated by filtering white noise through a minimum phase filter, spectral and residual analysis can be used for validation of linear models of nonlinear systems. Furthermore, it can be shown that the input minimum phase property guarantees that a certain optimality result will hold. Here, the benefits of using minimum phase instead of non-minimum phase filters for input design will be shown both theoretically and in numerical experiments.

1. Introduction

In this paper, we will discuss some properties of linear model estimates obtained by system identification using input and output data from nonlinear systems. The system identification method that will be studied here is the *prediction-error method* [6].

Consider a parameterized stable linear time-invariant (LTI) output error (OE) model

$$y(t) = G(q, \theta)u(t) + e(t),$$

where θ is a parameter vector and where q denotes the shift operator, $qu(t) = u(t + 1)$. The signals $u(t)$, $y(t)$ and $e(t)$ are the system input, output and noise, respectively.

It can be shown [5] that the prediction-error parameter estimate under rather general conditions will converge to the parameters that minimize a *mean-square error criterion* $\mathrm{E}((y(t) - G(q, \theta)u(t))^2)$. Here, $\mathrm{E}(x)$ denotes the expected value of the random value x. This result motivates why it is interesting to study the mean-square error optimal LTI approximation of a general, possibly nonlinear, system.

LTI approximations of nonlinear systems have been studied for a long time. A couple of useful, classical results concerning static nonlinearities can be found in [1] and [11]. More recent results about LTI approximations in a stochastic framework can be found in [12], [13], [14] and [15]. LTI approximations of nonlinear systems have been studied in a deterministic framework in [8], [9] and [10]. Some results using the same framework as here can be found in [7], [2] and [3].

2. Output Error LTI-SOEs

The class of nonlinear systems studied in this paper is defined by the following assumptions on the input and output signals.

Assumption A1. Assume that

(i) The input $u(t)$ and output $y(t)$ are real-valued stationary stochastic processes with $\mathrm{E}(u(t)) = \mathrm{E}(y(t)) = 0$.

(ii) There exist $K > 0$ and α, $0 < \alpha < 1$ such that the second order moments $R_u(\tau) = \mathrm{E}(u(t)u(t - \tau))$, $R_{yu}(\tau) = \mathrm{E}(y(t)u(t-\tau))$ and $R_y(\tau) = \mathrm{E}(y(t)y(t-\tau))$ satisfy $|R_u(\tau)| < K\alpha^{|\tau|}$, $|R_{yu}(\tau)| < K\alpha^{|\tau|}$ and $|R_y(\tau)| < K\alpha^{|\tau|}$ for all $\tau \in \mathbb{Z}$.

(iii) The z-spectrum $\Phi_u(z)$ (i.e., the z-transform of $R_u(\tau)$) has a canonical spectral factorization

$$\Phi_u(z) = L(z)r_u L(z^{-1}), \tag{1}$$

where $L(z)$ and $1/L(z)$ are causal transfer functions that are analytic in $\{z \in \mathbb{C} : |z| \geq 1\}$, $L(+\infty) = 1$ and r_u is a positive constant.

The type of LTI approximations studied here is described in the following definition.

Definition 2.1. Consider a nonlinear system with input $u(t)$ and output $y(t)$ such that Assumption A1 is fulfilled. The *Output Error LTI Second Order Equivalent* (OE-LTI-SOE) of this system is the stable and causal LTI model $G_{0,OE}(q)$ that minimizes the mean-square error $\mathrm{E}((y(t) - G(q)u(t))^2)$, i.e.,

$$G_{0,OE}(q) = \arg\min_{G\in\mathcal{G}} \mathrm{E}((y(t) - G(q)u(t))^2),$$

where $\mathcal{G}$ denotes the set of all stable and causal LTI models.

The following theorem is a direct consequence of classic Wiener filter theory.

Theorem 2.1
Consider a nonlinear system with input $u(t)$ and output $y(t)$ such that Assumption A1 is fulfilled. Then the OE-LTI-SOE $G_{0,OE}$ of this system is

$$G_{0,OE}(z) = \frac{1}{r_u L(z)} \left[\frac{\Phi_{yu}(z)}{L(z^{-1})} \right]_{\text{causal}}, \tag{2}$$

where $[\ldots]_{\text{causal}}$ denotes taking the causal part and where $L(z)$ is the canonical spectral factor of $\Phi_u(z)$ from (1).

Proof: See, for example, [6] or [4]. □

3. Minimum Phase Input Filters

A common way to generate a signal u such that its spectral density is equal to some predefined function is to filter white noise e through an LTI filter $L(z)$. If u is going to be used as input to an LTI system in a linear identification experiment, such a filter can be designed without considering its phase. However, if the signal u is to be used for an LTI approximation of a nonlinear system, the phase of the input filter is crucial for the behavior of this approximation. A fundamental property of OE-LTI-SOEs for minimum phase input filters is shown in the following theorem.

Theorem 3.1
Consider a causal nonlinear system with input $u(t)$ and output $y(t)$ such that Assumption A1 is fulfilled. Assume that the input signal has been generated by filtering white, possibly non-Gaussian, noise $e(t)$ through a minimum phase filter $L_m(z)$. Assume also that any other external signals that affect the output are independent of u. Then the OE-LTI-SOE is

$$G_{0,OE}(z) = \frac{\Phi_{yu}(z)}{\Phi_u(z)} = \frac{\Phi_{ye}(z)}{L_m(z)R_e(0)}. \tag{3}$$

Proof: The canonical spectral factorization of $\Phi_u(z)$ is $L(z) = \frac{L_m(z)}{l_m(0)}$, $r_u = l_m(0)^2 R_e(0)$. Using (2) from Theorem 2.1, this gives

$$\begin{aligned} G_{0,OE}(z) &= \frac{l_m(0)}{l_m(0)^2 R_e(0) L_m(z)} \left[\frac{l_m(0)\Phi_{yu}(z)}{L_m(z^{-1})} \right]_{\text{causal}} \\ &= \frac{1}{R_e(0)L_m(z)} \left[\frac{\Phi_{ye}(z)L_m(z^{-1})}{L_m(z^{-1})} \right]_{\text{causal}} \\ &= \frac{1}{R_e(0)L_m(z)} \left[\Phi_{ye}(z) \right]_{\text{causal}}. \end{aligned}$$

By the assumptions, the nonlinear system is causal and u is independent of all other external signals that affect the output. This, together with the fact that e is a white noise process, imply that $y(t)$ is independent of $e(t-\tau)$ for all $\tau < 0$. Hence, $R_{ye}(\tau) = 0$ for all $\tau < 0$ and

$$\Phi_{ye}(z) = \sum_{\tau=0}^{\infty} R_{ye}(\tau) z^{-\tau}.$$

Since the series $\Phi_{ye}(z)$ contains no positive powers of z, taking the causal part does not remove anything. Hence, we have $\left[\Phi_{ye}(z)\right]_{\text{causal}} = \Phi_{ye}(z)$ and we have shown (3). □

OE-LTI-SOEs that can be written as the ratio between $\Phi_{yu}(z)$ and $\Phi_u(z)$ exhibit a couple of interesting properties. The perhaps most obvious property that holds in this case concerns the residuals $\eta_0(t)$. In the following lemma it will be shown that for such an OE-LTI-SOE, the residuals will be uncorrelated with *all* input signal components.

Lemma 3.1
Consider a nonlinear system with input $u(t)$ and output $y(t)$ such that Assumption A1 is fulfilled. Assume that the OE-LTI-SOE can be written as

$$G_{0,OE}(z) = \frac{\Phi_{yu}(z)}{\Phi_u(z)} \tag{4}$$

and let

$$\eta_0(t) = y(t) - G_{0,OE}(q)u(t). \tag{5}$$

Then it follows that

$$\Phi_{\eta_0 u}(z) = \Phi_{yu}(z) - G_{0,OE}(z)\Phi_u(z) = 0, \tag{6a}$$

$$\Phi_{\eta_0}(z) = \Phi_y(z) - G_{0,OE}(z)\Phi_u(z)G_{0,OE}(z^{-1}). \tag{6b}$$

Proof: The expression for $\Phi_{\eta_0 u}(z)$ in (6a) follows directly from (4) and (5). Furthermore, (4) and (5) also give

$$\begin{aligned} \Phi_{\eta_0}(z) &= \Phi_y(z) - G_{0,OE}(z)\Phi_{uy}(z) - \Phi_{yu}(z)G_{0,OE}(z^{-1}) \\ &\quad + G_{0,OE}(z)\Phi_u(z)G_{0,OE}(z^{-1}) \\ &= \Phi_y(z) - G_{0,OE}(z)\Phi_u(z)G_{0,OE}(z^{-1}) \end{aligned}$$

and hence (6b) has been shown. □

Intuitively, it seems that it should always be a good idea to use input signals for which the OE-LTI-SOE is equal to $\Phi_{yu}(z)/\Phi_u(z)$ since it can be shown that this ratio defines always the mean-square error optimal noncausal LTI model, i.e., the noncausal LTI-SOE. As a matter of fact, input signals for which the OE-LTI-SOE of a nonlinear system can be written as $G_{0,OE}(z) = \Phi_{yu}(z)/\Phi_u(z)$ exhibit the following optimality property.

Theorem 3.2
Consider a nonlinear system with input $u_1(t)$ and output $y_1(t)$ such that Assumption A1 is fulfilled. Let $G_{0,OE,1}(z)$ denote the OE-LTI-SOE of the nonlinear system with respect to u_1 and assume that it can be written as

$$G_{0,OE,1}(z) = \frac{\Phi_{y_1 u_1}(z)}{\Phi_{u_1}(z)}.$$

Furthermore, let $\eta_{0,1}(t) = y_1(t) - G_{0,OE,1}(q)u_1(t)$.

Consider also another input signal $u_2(t)$ to the same nonlinear system. Assume that this signal generates the output $y_2(t)$ and that $(u_2(t), y_2(t))$ satisfy Assumption A1. Let $G_{0,OE,2}(z)$ denote the OE-LTI-SOE of the nonlinear system *with respect to u_2 and let $\eta_{0,2}(t) = y_2(t) - G_{0,OE,2}(q)u_2(t)$. Assume that*

$$\Phi_{u_2}(e^{i\omega}) = \Phi_{u_1}(e^{i\omega}), \quad \forall\omega \in [-\pi, \pi],$$
$$|\Phi_{y_2u_2}(e^{i\omega})| = |\Phi_{y_1u_1}(e^{i\omega})|, \quad \forall\omega \in [-\pi, \pi],$$
$$R_{y_2}(0) = R_{y_1}(0).$$

Then, the model residual variance for the OE-LTI-SOE corresponding to u_2 cannot be smaller than it is for the one corresponding to u_1, i.e.,

$$R_{\eta_{0,2}}(0) \geq R_{\eta_{0,1}}(0). \tag{7}$$

Proof: It can be shown [2] that for any OE-LTI-SOE we have

$$R_{\eta_0}(0) = R_y(0) - \frac{1}{2\pi}\int_{-\pi}^{\pi} |G_{0,OE}(e^{i\omega})|^2 \Phi_u(e^{i\omega})d\omega. \tag{8}$$

For any input signal, the noncausal LTI-SOE is always $\frac{\Phi_{yu}(z)}{\Phi_u(z)}$. It is easy to verify that (8) holds also for the noncausal LTI-SOE if $G_{0,OE}(e^{i\omega})$ is replaced by $\frac{\Phi_{yu}(e^{i\omega})}{\Phi_u(e^{i\omega})}$. As the stable and causal LTI systems are a subset of the stable and noncausal, it follows that the OE-LTI-SOE will always have a minimum mean-square error that is greater than or equal to the minimum mean-square error that is obtained for the noncausal LTI-SOE. Hence,

$$\begin{aligned} R_{\eta_{0,2}}(0) &\geq R_{y_2}(0) - \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{\Phi_{y_2u_2}(e^{i\omega})}{\Phi_{u_2}(e^{i\omega})}\right|^2 \Phi_{u_2}(e^{i\omega})d\omega \\ &= R_{y_1}(0) - \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{\Phi_{y_1u_1}(e^{i\omega})}{\Phi_{u_1}(e^{i\omega})}\right|^2 \Phi_{u_1}(e^{i\omega})d\omega \\ &= R_{\eta_{0,1}}(0) \end{aligned}$$

since $G_{0,OE,1}(z) = \frac{\Phi_{y_1u_1}(z)}{\Phi_{u_1}(z)}$. □

A common way to validate an estimated model of an open-loop LTI system is to compare the frequency response of the model with a nonparametric frequency response estimate obtained by spectral analysis. If these frequency responses are similar this indicates that the order of the parametric model is sufficiently high and that the numerical computation of the estimate has been successful. In Ljung [6, Sec. 6.4] it is shown that the spectral analysis frequency response estimate $\hat{G}_N(e^{i\omega_0})$ based on N measurements can be written

$$\hat{G}_N(e^{i\omega_0}) = \frac{\hat{\Phi}_{yu}^N(e^{i\omega_0})}{\hat{\Phi}_u^N(e^{i\omega_0})},$$

where $\hat{\Phi}_u^N(e^{i\omega_0})$ and $\hat{\Phi}_{yu}^N(e^{i\omega_0})$ are estimates of the spectral and cross-spectral densities that can be written as smoothed periodograms.

If an LTI model is estimated for an open-loop nonlinear system, it might be tempting to use spectral analysis as a validation method also in this case. However, the spectral analysis frequency response estimate can be quite different from the frequency response of the OE-LTI-SOE and is thus in general useless for validation purposes. Only when the OE-LTI-SOE can be written as $G_{0,OE}(z) = \Phi_{yu}(z)/\Phi_u(z)$, spectral analysis can be used as a validation method.

Example 3.1

Consider the simple static nonlinear system

$$y(t) = u(t)^3 \tag{9}$$

and the two input signals

$$u_1(t) = e(t) + \frac{1}{2}e(t-1), \quad u_2(t) = \frac{1}{2}e(t) + e(t-1),$$

where $e(t)$ is a sequence of independent random variables with uniform distribution over the interval $[-1, 1]$.

Straightforward calculations show that the OE-LTI-SOEs of this system are

$$\frac{0.85 + 0.575z^{-1}}{1 + 0.5z^{-1}} \text{ and } \frac{0.925 + 0.425z^{-1}}{1 + 0.5z^{-1}},$$

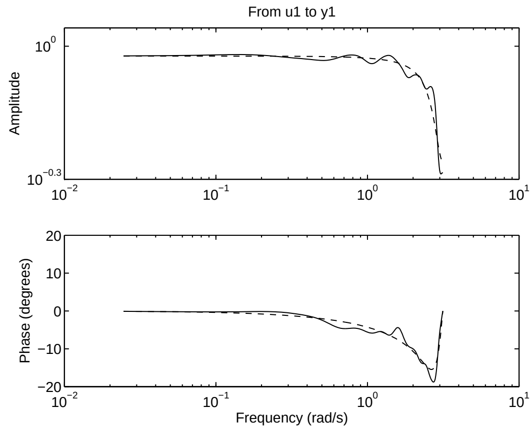
respectively. Furthermore, the two inputs and the corresponding outputs satisfy the conditions in Theorem 3.2.

Two data sets with 10000 noise-free input and output measurements have been generated. The first of these data sets was generated with a realization of the minimum phase filtered signal $u_1(t)$ as input while the second data set was generated with a realization of the non-minimum phase filtered signal $u_2(t)$ as input.

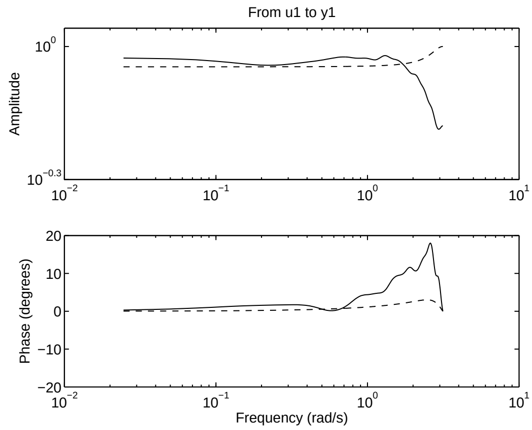
Nonparametric frequency response estimates have been computed from these data sets using spectral analysis with a Hamming window of lag size 30. These estimates are shown in Figure 1 together with the frequency responses of the corresponding OE-LTI-SOEs.

In Figure 1, it can be seen that there is a close match between the OE-LTI-SOE and the nonparametric frequency response estimate when the input has been generated by a minimum phase filter. However, when the input has been generated by a non-minimum phase filter, the OE-LTI-SOE is quite different from the nonparametric estimate.

The conclusion that can be drawn from the previous example is that for LTI approximations of nonlinear systems, spectral analysis can be used as a validation method only when an input signal that guarantees that $G_{0,OE}(z) = \Phi_{yu}(z)/\Phi_u(z)$ has been used. An additional property of such input signals is that they make the result of another validation method, residual analysis, easier to interpret since the residuals then by (6a) will be uncorrelated with all input components.



***(a)** The OE-LTI-SOE (dashed) and the spectral analysis estimate (solid) for an input generated by a minimum phase filter.*



***(b)** The OE-LTI-SOE (dashed) and the spectral analysis estimate (solid) for an input generated by a non-minimum phase filter.*

***Figure 1:** A nonparametric frequency response estimate will be a good approximation of the OE-LTI-SOE only when $G_{0,OE}(z) = \Phi_{yu}(z)/\Phi_u(z)$.*

4. Conclusions

In this paper, it has been shown that it is beneficial to use an input with the minimum phase property when approximating a nonlinear system with an LTI model. With such an input, spectral and residual analysis can be used as validation methods like for LTI systems.

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