2005 International Symposium on Nonlinear Theory and its Applications (NOLTA2005)
Bruges, Belgium, October 18-21, 2005

Nonlinear State Space Modelling Of Multivariable Systems

J. Paduart, J. Schoukens, R. Pintelon

Vrije Universiteit Brussel, dep. ELEC, Pleinlaan 2, 1050 Brussels, Belgium
Email: johan.paduart@vub.ac.be

Abstract - In this paper we propose a method to model nonlinear multivariable systems. We will use a state space approach since this is inherently compatible with Multiple Input, Multiple Output (MIMO) systems. The basic idea is to fit first a linear model on the measured data, and to extend that model so that it can grasp the nonlinear behaviour of the system. The results are applied to a physical system.

1. Introduction

Recently in the area of system identification, there has been an increasing interest in the modelling of nonlinear systems. This is because most real-life systems can be modelled quite well with a linear model, but even better results can be obtained with a nonlinear model. In this work, we want to focus on the identification of multivariable systems, using a state space approach. Earlier work in this area includes Local Linear Models, Linear Parameter Varying Systems or bilinear models ([3]). In this paper we use a different approach: first, a linear state space model is fit through the Best Linear Approximation of the system. Then, the linear model is extended in order to capture the nonlinear behaviour.

The structure of the paper is the following: first, we explain the identification procedure. Then, the identification method is applied on data from a physical nonlinear system for the SISO case and on simulated data for the MIMO case.

2. Model and Identification

The classical discrete time linear state equation is given by:

$$\begin{aligned} \boldsymbol{x}(k+1) &= A\boldsymbol{x}(k) + B\boldsymbol{u}(k) \\ \boldsymbol{y}(k) &= C\boldsymbol{x}(k) + D\boldsymbol{u}(k) \end{aligned} \tag{1}$$

where $\boldsymbol{x}(k)$ is the (n x 1) dimensional state vector, $\boldsymbol{u}(k)$ the (m x 1) input vector, $\boldsymbol{y}(k)$ the (p x 1) output vector, and A, B, C, and D are the system matrices. We propose to extend this model to:

$$\begin{aligned} \boldsymbol{x}(k+1) &= A\boldsymbol{x}(k) + B\boldsymbol{u}(k) + NL\boldsymbol{w}(k) \\ \boldsymbol{y}(k) &= C\boldsymbol{x}(k) + D\boldsymbol{u}(k) \end{aligned} \tag{2}$$

where the column vector $\boldsymbol{w}(k)$ contains static nonlinear terms, composed by cross products between the elements of the state vector. All possible product combinations between state elements for a chosen set of degrees are formed. The coefficients present in matrices A, B, C, D and NL are to be determined by the identification procedure described below.

For example, for order $n = 2$, degree $r = 2{:}3$, we have:

$$\boldsymbol{w}(k) = \begin{bmatrix} x_1^2(k) & x_1(k)\cdot x_2(k) & x_2^2(k) & \dots \\ x_1^3(k) & x_1^2(k)\cdot x_2(k) & x_1(k)\cdot x_2^2(k) & x_2^3(k) \end{bmatrix}^T \tag{3}$$

If a similarity transformation $\tilde{\boldsymbol{x}}(k) = T^{-1}\boldsymbol{x}(k)$ is applied to the state vector, the classical relationships ($\tilde{A} = T^{-1}AT$, $\tilde{B} = T^{-1}B$ and $\tilde{C} = CT$) still hold. The relationship between $\tilde{NL}$ and NL is not so trivial, but can be computed easily by substitution.

The first step in the identification procedure is to determine the Best Linear Approximation of the system. This can be done in an easy way by utilizing specially designed periodical excitation signals, namely Multisines. For these signals, which are composed of sines at selected frequencies, a fully custom amplitude spectrum can be set. By applying different random phase realisations and averaging the measured data, one can calculate the Best Linear Approximation of the system ([5], [6], [7], [8]). One can also use a single realisation to determine a linear model of less good quality. Then a subspace identification technique is used to fit a linear, discrete time, state space model. Since periodic excitation signals are used to excite the system, the identification can be performed easily in the frequency domain ([1], [2]).

The linear state space model of the Best Linear

This work was supported by the FWO-Vlaanderen, the Flemish Community (Concerted action ILiNoS), and the Belgian Government (IUAP-V/22).

Approximation (BLA) is then given by the following set of equations:

$$\begin{aligned} \boldsymbol{x}(k+1) &= A_{BLA}\boldsymbol{x}(k) + B_{BLA}\boldsymbol{u}(k) \\ \boldsymbol{y}(k) &= C_{BLA}\boldsymbol{x}(k) + D_{BLA}\boldsymbol{u}(k) \end{aligned} \tag{4}$$

The aim now is to determine system matrices A, B, C, D and NL. We define the following cost function:

$$K = \sum_{i=1}^{p}\sum_{k=1}^{N} \left(y_{i,\,\mathrm{meas}}(k) - y_{i,\,\mathrm{model}}(k)\right)^2 \tag{5}$$

The minimisation of this cost function is a problem that is nonlinear in the parameters. It can be solved iteratively in the time domain, for instance with a Levenberg-Marquardt routine for which starting values are needed. For A, B, C and D we can use the matrices obtained from the Best Linear Approximation. The starting value for NL is set to zero which implies that the iteration starts from the Best Linear Approximation and that our nonlinear model will perform at least as good as that model.

To compute the Jacobian, one also needs starting values for the states. This can be seen in the following example, where we compute the Jacobian for the elements of the system matrix A:

$$\boldsymbol{J}_{A_{ij}}(k) = \frac{\partial}{\partial A_{ij}}\boldsymbol{y}(k) = C\frac{\partial}{\partial A_{ij}}\boldsymbol{x}(k) \tag{6}$$

where element (i,j) of A is denoted as A_{ij}. If we define:

$$\boldsymbol{Jx}_{A_{ij}}(k) = \frac{\partial}{\partial A_{ij}}\boldsymbol{x}(k) \tag{7}$$

then it follows that:

$$\begin{aligned} \boldsymbol{Jx}_{A_{ij}}(k) = \frac{\partial}{\partial A_{ij}}A\cdot x(k-1) + A\cdot \boldsymbol{Jx}_{A_{ij}}(k-1) + \ldots \\ NL\cdot\frac{\partial}{\partial A_{ij}}\boldsymbol{w}(k-1) \end{aligned} \tag{8}$$

The expressions for $\boldsymbol{J}_{B_{ij}}(k)$, $\boldsymbol{J}_{C_{ij}}(k)$, $\boldsymbol{J}_{D_{ij}}(k)$ and $\boldsymbol{J}_{NL_{ij}}(k)$ are computed in the same way. From these expressions, we conclude the following:

1. The calculation of the Jacobian has to be performed recursively: to calculate $\boldsymbol{Jx}_{A_{ij}}(k)$, $\boldsymbol{Jx}_{A_{ij}}(k-1)$ is needed.
2. One also needs estimates of the states $\boldsymbol{x}(k-1)$. To obtain these, the calculated states from the previous iteration in the Levenberg-Marquardt loop are used.

Another issue that needs to be addressed is the rank deficiency of the Jacobian J. This deficiency exists due to the non-uniqueness of the state space representation. As mentioned before, a similarity transformation $\tilde{\boldsymbol{x}}(k) = T^{-1}\boldsymbol{x}(k)$ leaves the input-output behaviour unaffected. As a consequence, the Jacobian J will not be of full rank. This can be taken care of by using a truncated Singular Value Decomposition (SVD) when computing the pseudo inverse of J ([4]).

3. Results

3.1. SISO

The method described in the previous paragraph is applied to an electrical circuit that simulates the behaviour of a mass-spring-damper system with a nonlinear spring (this circuit is also known as the "Silverbox"). The estimation data set consists of 8 consecutive realisations (each 8192 samples) of a multisine. For validation purposes, a sequence of white noise (40 700 data points) has been applied to the system, with an increasing amplitude. At the end of the sequence, the amplitude exceeds that of the multisines from the estimation data set, so that the extrapolation behaviour of the model can be investigated (Fig. 1.).

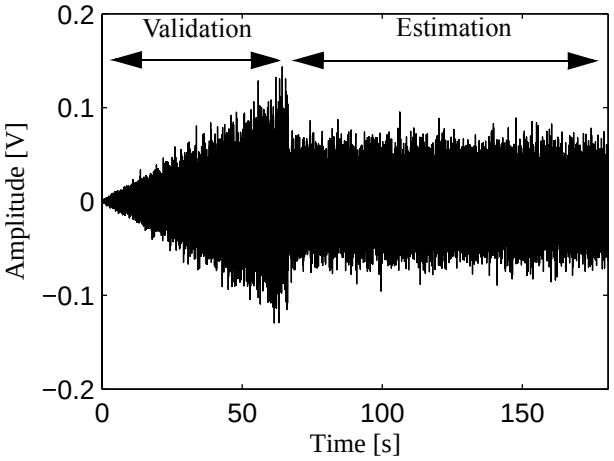

Fig. 1. Estimation/Validation data set

The first step is to average the measured FRF's of the 8 multisines, which results in the Best Linear Approximation. A linear model (order $n = 2$) is fitted and validated (Fig. 2., Fig. 3.) It is clear that towards larger amplitudes, the quality of the linear model drops significantly.

The linear model is extended as described in the previous section (degree $r = 2{:}3$). After 30 iterations of the Levenberg-Marquardt loop, the cost function does not

decrease significantly any more. We can now compare the errors of both models. For the calculation of the Root Mean Square (RMS) value of the error signals, the first 200 samples have been discarded in order to eliminate the transient behaviour. The RMS level of the error has dropped with a factor of 25 (30 dB) from 14.4 mV for the linear model to 0.57 mV for the nonlinear model.

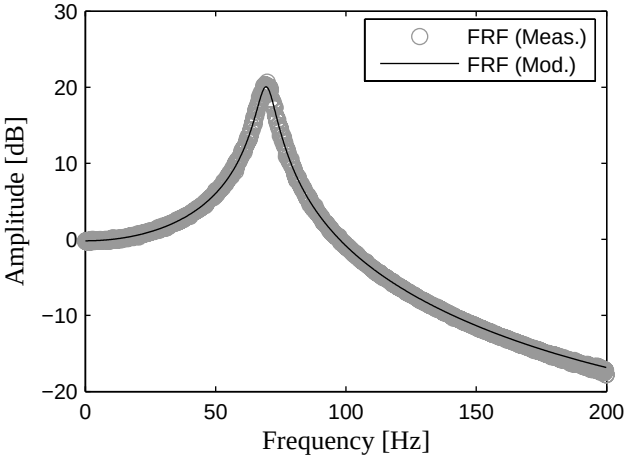

Fig. 2. Averaged Frequency Response Function

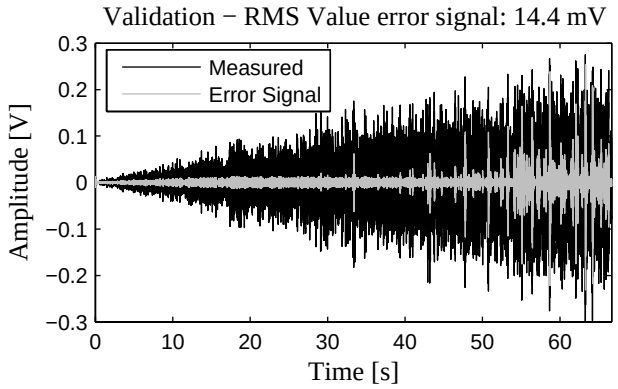

Fig. 3. Validation test linear model

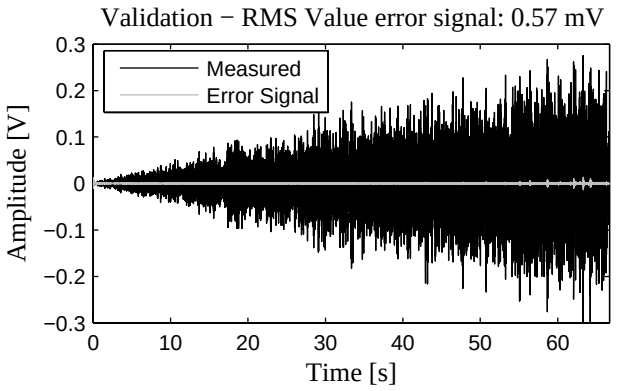

Fig. 4. Validation test nonlinear model

3.2. MIMO

The MIMO case will be illustrated by means of a simulation. We will try to model a nonlinear system with two inputs and two outputs, with the following topology:

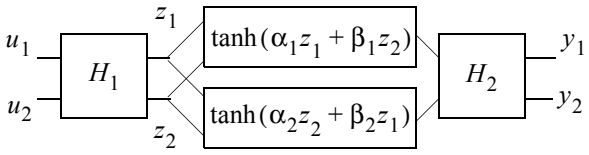

Fig. 5. Nonlinear MIMO System

The blocks H_1 and H_2 are discrete time, linear, dual input, dual output systems, with the following parameters:

$$A_1 = \begin{bmatrix} 0.2779 & -0.4152 \\ 0.4152 & 0.8651 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0.1381 & -0.4714 \\ 0.4714 & 0.8047 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0.5872 & 0.6316 \\ 0.1908 & 0.1980 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0.6667 & 0.5770 \\ 0.2761 & 0.3171 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0.1468 & 0.6594 \\ 0.1473 & 0.7827 \end{bmatrix} \quad C_2 = \begin{bmatrix} 0.1667 & 0.6381 \\ 0.1465 & 0.5881 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} -0.0767 & -0.0977 \\ -0.0107 & -0.0964 \end{bmatrix} \quad D_2 = \begin{bmatrix} -0.2379 & 0.0257 \\ -0.0838 & -0.0184 \end{bmatrix}$$

The two intermediate blocks with hyperbolic tangents represent saturation and intermixing behaviour. The parameters of these nonlinear blocks were chosen as follows:

$$\alpha_1 = \alpha_2 = 5$$
$$\beta_1 = \beta_2 = 0.5$$

This system could be interpreted as a differential amplifier (with low pass characteristics) which shows some saturation behaviour and mixing of the two channels. To identify this system, a single realisation of a multisine is applied to both inputs at the same time. After removal of the transients, a linear model is fitted (n = 4), and the Levenberg-Marquardt loop is started (degree r = 2:5). To validate the identified model parameters, a second realisation of a multisine is applied to the Device Under Test. The process of identification and validation is performed 20 times (each with a new set of multisine realisations), and the mean values of the error relative RMS (Root Mean Square) values are given in Table 1. The model error is reduced with a factor of about 40 (30 dB) in comparison with a linear model. In fact, the error of the nonlinear model can be arbitrarily decreased by increasing

Table I: Relative RMS Error

Model	y_1	y_2
Linear	21.0 mV	21.2 mV
Nonlinear	0.55 mV	0.53 mV

the degree r, because the hyperbolic tangent function is a function that needs an infinite number of polynomial terms when expanded in a Taylor series.

4. Conclusions

In this paper we have introduced a new method to model nonlinear multivariable systems. The model performs quite well on measured data as well as in simulations. Nevertheless, some issues still need to be addressed: a stability criterion needs to be developed, as well as a regularisation procedure to reduce the number of parameters as the system order n becomes large.

REFERENCES

[1] McKelvey, T., Akçay, H., Ljung, L. (1996). Subspace-based multivariable system identification from frequency response data. *IEEE Transactions on Automatic Control, 41(7),* pp. 960-979.

[2] R. Pintelon (2002). Frequency-domain subspace system identification using non-parametric noise models. *Automatica 38,* pp. 1295-1311.

[3] V. Verdult (2002). Nonlinear System Identification - A State Space Approach, *Twente University Press*

[4] G. H. Golub, C. F. Van Loan (1996). *Matrix Computations - Third Edition.*The John Hopkins University Press, London.

[5] R. Pintelon, J. Schoukens (2001). System Identification. A Frequency domain approach. *IEEE Press, New Jersey.*

[6] T. Dobrowiecki, J. Schoukens (2004). Linear Approximation of Weakly Nonlinear MIMO Systems. *Proceedings of IMTC 2004, p. 1607-1612.*

[7] J. Schoukens, T. Dobrowiecki, and R. Pintelon (1998). Identification of linear systems in the presence of nonlinear distortions. A frequency domain approach. *IEEE Trans. on Automatic Control,* **Vol. 43**, No.2, pp. 176-190.

[8] J. Schoukens, J. Swevers, R. Pintelon, H. Van der Auweraer (2002). Excitation design for FRF measurements in the presence of nonlinear distortions. *Proceedings of ISMA2002*, **Vol. 2**, 951-958.