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Efficiency and Robustness in Proportion Regulation Systems

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Abstract—Designable proportion regulation systems which consist of identical agents using stochastic automata are suggested. From the viewpoint of the group response and the individual behavior, the performances of a simple model and an improved one are compared numerically.

1. Introduction

Social insects such as bees or ants exhibit *polyethism*, i.e., a system for division of labor[1]. In a colony, several kinds of tasks — nursey, nest maintaining, foraging and so on — are allocated for numerous individuals of that colony mainly by their age. This intriguing phenomenon has attracted many scientists of various fields, such as biology, chemistry and physics. Problem concerning to the mechanism for the system is not fully understood up to now. Recent studies reveal that the diversity of internal states (or agents) improves the efficiency of the colony as a whole [2]. As another example, the differentiation phenomena of the cellular slime molds can be regarded as a kind of task allocation of genetically identical cells with an appropriate proportion between different cell types [3]. In a society of human beings, self organization of the division of labor without external orders is sometimes experienced in our daily life.

The task allocation with the proportion regulation has many merits; The system adapts itself to the variations of the environment autonomously, it does not require sophisticated information processing abilities to each individual, high efficiencies are expected by learning of each individual, the whole system is robust against the disturbance such as loss of a part of individuals. For these reasons, it is worth studying this system from the viewpoint of science and engineering.

In this paper, we suggest designable proportion regulation systems using stochastic automata and compare them. We consider that proportion regulation system has following properties: (i) it consists of a mass of (almost) identical individuals, (ii) they divide into several states, (iii) and the proportions of population between the states are regulated in a certain range against various disturbances. (iv) Supervising individuals are not necessary.

2. Stochastic Automata Model

Recently, several models of proportion regulation system have been investigated; threshold model for task allocation [4], global Türing model [5] and variable potential models [6]. We here focus on a *stochastic automata* model to design the proportion regulation systems for the various architecture of the states (see Fig. 1).

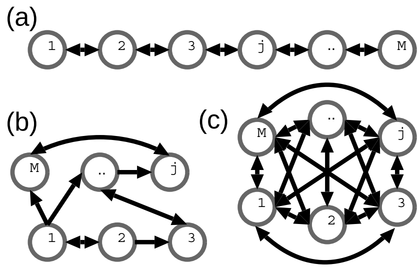

Figure 1: Several architecture of the states and the rule, sequential (a), general (b) and all-to-all (c) types. Gray circles and black arrows represent the states and the transition rule between them, respectively.

Let the total number of individuals of the system be N and the number of states be M. Each individual has a set of transition probability between the states $\{p_{ij}\}$ and it generates a random number s iteratively. The state at the next time step is determined by the value s and the transition rule using $\{p_{ij}\}$. For the simplicity, the random number s is uniform in the interval $[0, 1]$ without any temporal correlation. The transition probability between the states p_{ij} satisfies the condition $0 \le p_{ij} \le 1$, $i, j = 1, ..., M$, and the normalization condition $\sum_j p_{ij} = 1$. As a transition rule, we adopt following rule for the individuals in the i-th state (see also Fig. 2):

$$\sum_{j=1}^{k-1} p_{ij} \le s < \sum_{j=1}^{k} p_{ij} \Rightarrow \text{ transition to } k\text{-th state.} \quad (1)$$

Taking the continuum limit concerning to time, an evolution equation for the number of individual $n_i(t)$ in the i-th state at time t is represented as

$$\dot{\vec{n}} = F\vec{n}, \quad (2)$$

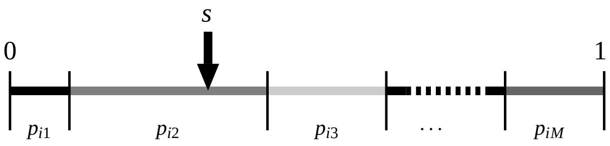

Figure 2: Schematic view of the transition rule for individuals in i-th state. At the next time step, an individual with value s jumps to state 2 in this case.

where $\vec{n} \equiv \begin{pmatrix} n_1 \\ \vdots \\ n_M \end{pmatrix}$ is a state vector and

$$(F)_{ij} \equiv \begin{cases} p_{ji} & i \neq j, \\ -\sum_{k \neq i} p_{ik} & i = j, \end{cases} \tag{3}$$

is an evolution matrix. This system is $M-1$ dimensional linear dynamical system considering the constraint $\sum n_i = N$. If we represent a proportion regulated steady state $\vec{n}^*$, it corresponds to a null eigenvector of F, i.e.,

$$\vec{0} = F\vec{n}^*. \tag{4}$$

Next, let a set of proportion between the designed states be

$$\{r_i^*\} \equiv (r_1^*, ..., r_M^*) \equiv \left(\frac{n_i^*}{N}, ..., \frac{n_M^*}{N}\right). \tag{5}$$

The number of the condition to design $\{r_i^*\}$ equals to $M-1$ while the number of $\{p_{ij}\}$ is $M(M-1)$. Therefore, there remains $(M-1)^2$ "degrees of freedom" of $\{p_{ij}\}$ even $\{r_i^*\}$ is specified. Using these remained degrees of freedom, we can design other aspects such as time scale or flow between the states with keeping the designed proportion.

For example, if $M = 3$ as depicted in Fig. 3, the transition matrix is

$$\begin{pmatrix} 1 - p_{12} - p_{13} & p_{12} & p_{13} \\ p_{21} & 1 - p_{21} - p_{23} & p_{23} \\ p_{31} & p_{32} & 1 - p_{31} - p_{32} \end{pmatrix}, \tag{6}$$

and the evolution matrix becomes

$$F = \begin{pmatrix} -p_{12} - p_{13} & p_{21} & p_{31} \\ p_{12} & -p_{21} - p_{23} & p_{32} \\ p_{13} & p_{23} & -p_{31} - p_{32} \end{pmatrix}. \tag{7}$$

Concerning to the steady state vector $\begin{pmatrix} n_1^* \\ n_2^* \\ n_3^* \end{pmatrix}$, the proportion between the states is determined only by the ratio of

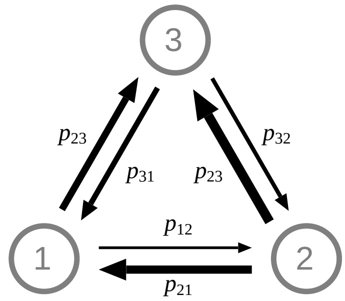

Figure 3: Example case of $M = 3$.

transition probabilities: $r_1^* : r_2^* : r_3^* = p_{32}p_{21} + p_{23}p_{31} + p_{21}p_{31} : p_{13}p_{32} + p_{31}p_{12} + p_{12}p_{32} : p_{12}p_{23} + p_{21}p_{13} + p_{13}p_{23}$. In this case, the total number of parameters p_{ij} equals six and the number of condition to determine the designed proportion is two. So there are four degrees of freedom to design other aspects of the system. One simple example is that if we multiply each of p_{ij} equally (keeping the condition $0 \leq p_{ij} \leq 1$), the designed proportion is unchanged and only the time scale varies. This is because the proportion r_j^* is given in the form of the homogeneous expression of p_{ij}. As other examples, several restricted rules are designable such as directed loops by choosing $p_{12} = p_{23} = p_{31} = 0$ (only "clockwise" transitions are allowed in Fig. 3). Note that symmetric transition probabilities ($p_{ij} = p_{ji}$) causes an equal proportion ($r_i^* = 1/M$).

3. Variable Probability Model

Individuals of the simple model described in the previous section is independent each other and there is no "synergic" mechanism. In this section, we try to control the performance of the system by introducing an interaction between the individuals.

First of all, in order to represent the deviation from a designed state, we introduce *stock materials* for each states. These materials represent a scale of sufficiency of each state. If we consider the social insects world, they may correspond to pheromones or room which is not occupied by the dust in the nest. Hereinafter, let a quantity of stock material of j-th states be w_j. Next, we represent a quantity of stock material of designed proportion w_j^*. We define basic transition probability $\{p_{ij}^0\}$ as the same value introduced in the previous section, i.e., $\{p_{ij}\}$ which gives $\vec{n}^*$. Using these quantities, we suggest a dynamical modification of the transition probability as follows:

$$\tilde{p}_{ij} = f(w_i - w_i^*) \times p_{ij}^0 \times g(w_j - w_j^*), \tag{8}$$

with normalization condition $p_{ij} = \tilde{p}_{ij} / \sum_j \tilde{p}_{ij}$. The functions $f(w)$ and $g(w)$ are modifier functions which represent the situations of the pre- and post-transition states and they satisfy following conditions: $f(w), g(w) > 0$, $f(0) = g(0) = 1$. As for a dynamics of the stock materials w_j,

$$\gamma_j \dot{w}_j = -w_j + \alpha_j n_j - \beta_j N, \qquad (9)$$

is considered. Here, the parameters α_j, β_j and γ_j are the production rate, the consumption rate and the decay rate of the stock materials, respectively. For the steady states $\dot{w}_j = 0$, $w_j = \alpha_j N(r_j - \beta_j/\alpha_j)$. The quantity of the stock material for the designed proportion r_j^* is then given as $w_j^* = \alpha_j N(r_j^* - \beta_j/\alpha_j)$. By assuming $\beta_j/\alpha_j = r_j^*$, we set $w_j^* = 0$.

4. Numerical Results

Here, we perform numerical simulations and compare the behavior of the two models. The number of states $M = 3$ and the transition probability $\{p_{ij}^0\}$ is set to $p_{12}^0 = p_{23}^0 = p_{31}^0 = 0$, $p_{13}^0 = 0.6\bar{p}$, $p_{21}^0 = 0.3\bar{p}$ and $p_{32}^0 = 0.2\bar{p}$ in order to design the proportion $r_1 : r_2 : r_3 = 1 : 2 : 3$ with the "clockwise" rule. $\bar{p}$ is a constant parameter which only changes the time scale. As the modifier functions, we adopt $f(w) = 1$ and $g(w) = \exp(-kw)$. If $k = 0$, it represents the simple (constant) probability model. To observe the response to the unsteady environment, the simulation is performed under several kinds of perturbation. Figure 4 shows typical time series in the two different environment, i.e., unperturbed ($-5000 < t < 0$) and perturbed ($0 < t < 5000$). In the perturbed case, all the individuals in the state 2 is forcedly moved to the state 3 at randomly chosen timing (as indicated by the downward arrows in Fig. 4).

With the simple probability model (k=0), the designed proportion are achieved statistically and the proportion between the states fluctuates to some extent in the unperturbed environment. In the perturbed case, the proportions deviates from the designed value $1/6 : 1/3 : 1/2$. On the other hand, with the variable probability model (k=0.1), the designed proportion is realized almost constantly in both environments. Especially, quick recoveries are observed in the perturbed case.

In order to characterize these features quantitatively, we introduce an order parameter which describes the group behavior. We here adopt the accuracy of the proportion A_p which is defined by the hamming distance between the obtained proportion $r_j(t)$ and the designed one r_j^* :

$$A_p(t) = \frac{1}{N}\sum_{j=1}^{M}(r_j(t) - r_j^*)^2. \qquad (10)$$

Using A_p, two systems can be compared from the viewpoint of the accuracy to the designed proportion, i.e., the smaller A_p, the more accurate the proportion is. Figure 5 shows the $\langle A_p \rangle$ vs the basic transition probability $\bar{p}$ of the two models in the perturbed environment, where $\langle A_p \rangle$ denotes the time average of A_p. For the simple model (black

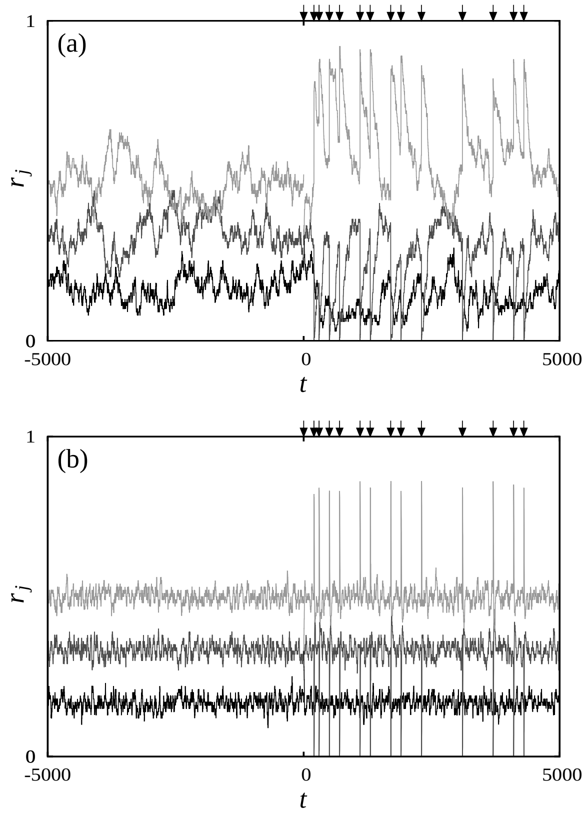

Figure 4: Typical time series of the proportion of each states with $M = 3$ and $N = 100$. Proportion is designed to $1 : 2 : 3$. (a) simple probability model ($k = 0$) and (b) variable probability model ($k = 0.1$). Black, dark gray, light gray lines show the proportion of the state 1,2 and 3, respectively. The system is unperturbed during $t < 0$ and in $t > 0$ the perturbations are added at the time denoted by the arrows.

line), $\langle A_p \rangle$ is a decreasing function of $\bar{p}$, which can be interpreted as the large transition probability enables the system to respond to the perturbation quickly. The variable model (gray line) realizes lower value of accuracy than the simple model, less than one of tenth in the low $\bar{p}$ regime. So, if we want to set the system more accurate in the unsteady environment, an increase of the transition probability by $\bar{p}$ or modifier $g(w)$ is effective.

Next, we focus on the behavior of individuals. As characteristic quantity, we introduce an average of individual resident time of the each state. Let τ_i^k is the k-th resident time of the i-th element as shown in Fig.6(a). The individual resident time is defined as

$$\tau_I \equiv \langle \tau_i^k \rangle, \qquad (11)$$

where $\langle\rangle$ denotes the average both on i and k. The smaller τ_I, the faster each individual transits between the states.

The reason why we see τ_I is that a kind of inefficiency is anticipated for too small τ_I by considering factors such

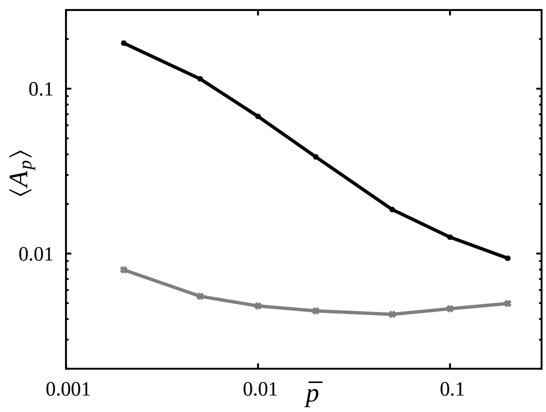

Figure 5: Averaged accuracy of the proportion $\langle A_p \rangle$ vs $\bar{p}$. Black and gray lines denote the simple model ($k = 0$) and the variable model ($k = 0.1$), respectively.

that the time loss required by changing state or learning. [1] Figure 6(b) shows the τ_I vs $\bar{p}$ for the simple model (black)

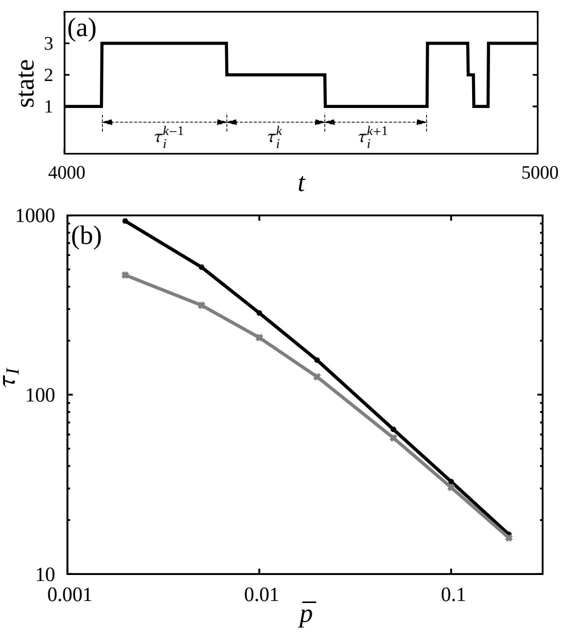

Figure 6: Typical evolution of i-th individual (a) and the average resident time τ_I vs $\bar{p}$ (b). Black and gray lines denote the simple model ($k = 0$) and the variable model ($k = 0.1$), respectively.

and the variable one (gray) in the perturbed environment. In both models, τ_I is a decreasing function of $\bar{p}$. The difference between the two models is not so large and the influence of the modifier function to the resident time is considered to be slight. The relative decrease of τ_I of the variable model in the region $\bar{p} < 0.01$ is the result of adaptation to the perturbed environments.

[1] If there is an effect of weariness — inefficiency caused by continuation of the same state, τ_I may be chosen within some extent.

Considering these two aspects represented by A_p and τ_I, we can choose the setting of the system depending on the situation. For example, if we want to increase both the accuracy and the individual resident time, the variable model with small $\bar{p}$ is suitable.

5. Discussion

Proportion regulation systems using stochastic automata are suggested. First, the simple probability model is introduced and it is shown that fully asymmetric system ($p_{ij} \neq p_{ji}$) has excess degrees of freedom to design the proportion of steady state $\vec{n}^*$. Second, the variable probability model is suggested to control the two quantitative aspects i.e., the accuracy A_p and the resident time of individual τ_I. Using these quantities, the performances of different systems can be compared. A reality of the stochastic transition rule in the biological correspondence, e.g., chemical reaction network, gene-metabolic network or neural network remains as an open problem.

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References

[1] E. O. Wilson, *The Insect Societies*, Oxford Univ. Press, London, 1971; *Sociobiology*, Harvard, Cambridge, MA, 1975.

[2] J. C. Jones, M. R. Myerscough, S. Graham and B. P. Oldroyd, *Science*, vol.305 pp.402–404, 2004.

[3] J. T. Bonner, *Q. Rev. Biol.* vol. 32 (1957) 232; *The Cellular Slime Molds*, Princeton Univ. Press, Princeton, New Jersey, 1967, 2nd ed.

[4] E. Bonabeau, M. Dorigo, G. Theraulaz, Swarm Intelligence, Oxford University press, Oxford, 1999.

[5] T. Mizuguchi, M. Sano, *Phys. Rev. Lett.*, vol.75, pp.966–969, 1995.

[6] T. Mizuguchi, et al., in preparation.