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Leading Parameters Controlling Synchronizability of Complex Networks

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Abstract—We study complete synchronization of chaotic oscillators in weighted complex networks and we show that the ability of the networks to achieve synchronization (synchronizability) is universally determined by two leading parameters in random networks and is independent of the degree distribution. The leading parameters are the mean degree and the heterogeneity of the distribution of node's intensity, where the intensity of a node, defined as the total strength of input connections, is a natural combination of topology and weights.

1. Introduction

Networks are playing an increasing role in the study of complex systems [1]. A problem of fundamental importance is the impact of network structures on the dynamics of the networks [2, 3, 4]. This problem has been recently also intensively studied in the context of synchronization of networks [5, 6, 7, 8, 9, 10]. Previous work on synchronization has focused mainly on the influence of the topology of the connections by assuming that the coupling strength is uniform. It has been shown that the ability of an oscillator network to synchronize depends critically on the average distance between nodes and the heterogeneity in the distribution of degrees (the number of connection of node) [9]. However, most complex networks where synchronization is relevant are actually weighted, for example the brain networks [11], networks of cities in the synchronization of epidemic outbreaks [12], and communication and other technological networks whose functioning relies on the synchronization of interacting units [13]. It has been shown that the weights of many real networks are often highly heterogeneous [14]. We have recently studied the effects of weighted coupling on synchronization [10] and found that the synchronizability is strongly influenced by the weight structure of the networks. A question still open is: Is there a general principle underlying the synchronization of networks with different topology and weight structure? And what are the leading parameters controlling the synchronizability of complex networks?

In this paper, we address this question in random networks with weighted coupling schemes motivated by real networks. We have found a universal formula that describes synchronizability of identical oscillators solely in terms of the mean degree and the heterogeneity of the node's intensity, irrespective of the degree distribution, clustering, degree correlation, and other topological properties. The intensity of a node, defined as the sum of the strengths of all input connections of that node, incorporates both topological and weighted properties and raises as a very important parameter controlling the synchronizability of complex networks.

The paper is organized as follows. In Sec. 2 we present the dynamical equation of the system and the measure of synchronizability. In Sec. 3 we discuss effects of a weighted coupling scheme motived from real networks, showing that the synchronizability is correlated with the heterogeneity of the intensity. With a general weighted coupling scheme, we demonstrate the universality of the synchronizability in Sec. 4. The universal formula is obtained in Sec. 5. Sec. 6 is devoted to the conclusion.

2. Dynamical Equation and Synchronizability

The dynamics of a general weighted network of N coupled identical oscillators is described by

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) + \sigma \sum_{j=1}^{N} W_{ij} A_{ij} [\mathbf{H}(\mathbf{x}_j) - \mathbf{H}(\mathbf{x}_i)], \quad (1)$$

$$= \mathbf{F}(\mathbf{x}_i) - \sigma \sum_{j=1}^{N} G_{ij} \mathbf{H}(\mathbf{x}_j), \quad i = 1, \ldots, N, \quad (2)$$

where $\mathbf{F} = \mathbf{F}(\mathbf{x})$ governs the dynamics of each individual oscillator, $\mathbf{H} = \mathbf{H}(\mathbf{x})$ is the output function, and σ is the overall coupling strength. Here $G = (G_{ij})$ is the coupling matrix combining both topology [adjacency matrix $A = (A_{ij})$] and weights [weight matrix $W = (W_{ij})$]: $G_{ij} = -W_{ij}$ for $i \neq j$ and $G_{ii} = \sum_j W_{ij} A_{ij} = S_i$. Here S_i denotes the intensity of node i, which is a significant measure integrating the information of connectivity and weights.

The linear stability of the synchronized state $\{\mathbf{x}_i = \mathbf{s}, \forall\ i\ |\dot{\mathbf{s}} = \mathbf{F}(\mathbf{s})\}$ can be assessed by diagonalizing the variational equations of Eq. (2) into N eigenmodes of the form

$$\dot{\xi}_i = [D\mathbf{F}(\mathbf{s}) - \sigma \lambda_i D\mathbf{H}(\mathbf{s})]\xi_i, \quad (3)$$

where λ_i denotes the ith eigenvalue of the coupling matrix G [15]. We assume that A is binary and symmetric, as in many previous studies [7, 6, 9], and we focus on the cases where G has real eigenvalues, $0 = \lambda_1 \leq \lambda_2 \cdots \leq \lambda_N$, with $\lambda_1 = 0$ corresponding to the eigenmode parallel to the synchronization manifold. For many oscillatory dynamical systems [7, 15], each transverse eigenmode is stable in a single, finite interval $\epsilon_1 < \sigma\lambda_i < \epsilon_2$, where the thresholds ϵ_1 and ϵ_2 are determined only by $\mathbf{F}$, $\mathbf{H}$, and $\mathbf{s}$. The network is thus synchronizable for some σ if the condition $\epsilon_1 < \sigma\lambda_i < \epsilon_2$ is satisfied for all $i \geq 2$, so that all the transverse eigenmodes are damped. This is equivalent to the condition

$$R \equiv \lambda_N/\lambda_2 < \epsilon_2/\epsilon_1, \tag{4}$$

where the eigenratio R depends only on the network structure, as defined by the coupling matrix G, and ϵ_2/ϵ_1 depends only on the dynamics. From these, it follows that the smaller the eigenratio R the more synchronizable the network and vice versa [7]. We then characterize the synchronizability of networks using the eigenratio R only.

3. Weighted Coupling on Synchronizability

The analysis of real networks, including scientific collaboration networks [14], metabolic networks [16], and airport networks [14, 16], has shown the following: (i) the weight W_{ij} of a connection between nodes i and j is strongly correlated with the product of the corresponding degrees (number of connections) as $\langle W_{ij}\rangle \sim (k_i k_j)^\theta$; (ii) the average intensity $S(k)$ of nodes with degree k increases as $S(k) \sim k^\beta$. A similar functional relationship between weights and network topology has been found in traffic-driven models where the weights are defined by the betweenness centrality [16] or link-load [17]. When the degree correlations can be neglected, the exponents in (i) and (ii) are related as $\beta = 1 + \theta$ [14]. In particular, $\theta \approx 0.5$ ($\beta \approx 1.5$) for the world-wide airport network, so that the intensities of the nodes grow faster than their degrees, while $\theta \approx 0$ ($\beta \approx 1$) for the cond-mat collaboration network [14]. The case $\theta < 0$ ($\beta < 1$) corresponds to a saturation in the capacity of nodes with large degrees and is expected to be relevant for other networks, such as neuronal and cortical networks.

Motivated by these observations from real networks, we consider the following weighted couplings,

$$W_{ij} = (k_i k_j)^\theta, \tag{5}$$

where θ is a tunable parameter. This model includes unweighted networks as special cases ($\theta = 0$ and $W_{ij} = 1$), which have been widely studied [7, 6, 9].

We now study the synchronizability of weighted networks in a model of growing scale-free networks (SFNs) with aging [18], which extends the Barabási-Albert (BA) model [19]. To build the networks, we start with m fully connected nodes and we add a new node with m links at each time step. The minimum degree is then $k_{\min} = m$ and the mean degree is $K = 2m$. The preferential attachment of new links is assumed to depend on the degree k_i and age τ_i of the corresponding node according to the probability $\Pi_i \sim k_i\tau_i^{-\alpha}$. The analysis in [18] shows that for the aging exponent $-\infty < \alpha \leq 0$, this growing rule generates SFNs with a power-law tail $P(k) \sim k^{-\gamma}$ and scaling exponent in the interval $2 < \gamma \leq 3$, as in most real SFNs. For $\alpha = 0$, we retain the usual BA model [19], which has $\gamma = 3$.

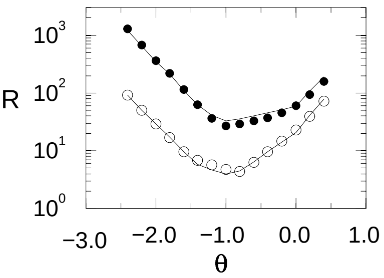

Figure 1: Eigenratio R as a function of θ in weighted growing SFNs with aging exponent $\alpha = 0$ (∘) and $\alpha = -3$ (•). Each symbol is an average over 50 realizations of the networks, for $K = 20$ and $N = 2^{10}$. The solid lines are the approximations of R by Eq. (10) with $A_R = 0.47$.

The weighted coupling scheme of Eq. (5) has a significant impact on the synchronizability of the networks. As θ is reduced from zero, the eigenratio R decreases and reaches a minimum around $\theta = -1$ for different networks [Fig. 1]. The eigenratio R is also shown as a function of the aging exponent α [Fig. 2(a)]. For unweighted networks ($\theta = 0$), the synchronizability decreases when α is reduced. When the networks are weighted with $\theta = -1$, the synchronizability is clearly enhanced. But which changes in the network underlie this enhanced synchronizability?

The degree distribution does not change with θ, and the individual weights W_{ij} become more heterogeneous when θ is reduced from $\theta = 0$. The intensities S_i of the nodes, however, become more homogeneous when $\theta \to -1$. This can be seen by $S_i = k_i^{1+\theta}\langle k_j^\theta\rangle_i$, where $\langle k_j^\theta\rangle_i = (1/k_i)\sum k_j^\theta$ is almost constant if the degree correlation can be neglected and $k_i \gg 1$, so that the intensity is almost homogeneous at $\theta = -1$. These observations suggest that the synchronizability of the networks is positively correlated with the homogeneity of S_i.

4. Universality

To investigate how the synchronizability depends on the intensity, we consider a more general weighted coupling model in which the node's intensities S_i follow an arbitrary distribution not necessarily correlated with the degrees k_i. In this model, the coupling weights are defined as

$$W_{ij} = S_i/k_i, \tag{6}$$

for all the k_i (input) connections of each node i. Note that the eigenvalues of the resulting coupling matrix G are still real. Now we analyze two different distributions of S_i which are uncorrelated with the distribution of k_i: (1) a uniform distribution in $[S_{\min}, S_{\max}]$; and (2) a power-law distribution, $P(S) \sim S^{-\Gamma}$, $S \geq S_{\min}$, where $S_{\min}$ is a

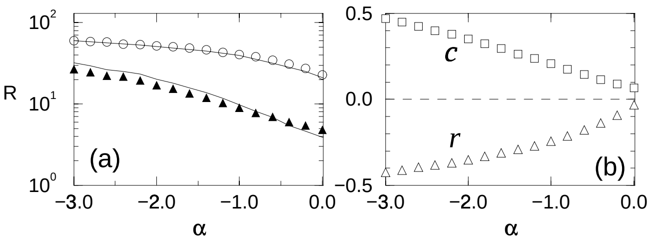

Figure 2: (a) Eigenratio R vs. α for growing SFNs with $\theta = 0$ (○) and $\theta = -1$. Solid lines: approximation by Eq. (10) with $A_R = 0.47$. (b) Clustering coefficient c (□), defined as in Ref. [1], and degree correlation coefficient r (△), defined as in Ref. [21], vs. α. The other parameters are the same as in Fig. 1.

positive number. In both cases, the parameter $S_{\max}/S_{\min}$ is taken as a measure of the heterogeneity of the distribution. We find that for a given mean degree K, the eigenratio R collapses into a single curve if plotted as a function of $S_{\max}/S_{\min}$ [Fig. 3(a)], irrespective of the distributions of k_i and S_i. These results provide strong evidence that the synchronizability is universally determined by the mean degree K and the heterogeneity of the intensities S_i. For a given K, the synchronizability is enhanced when S_i becomes more homogeneous.

5. Universal Formula and Leading Parameters

A more physical and quantitative understanding of this universality can be obtained by a mean field approximation of the dynamical system in Eq. (1). Let $\bar{\mathbf{H}}_i^W = (1/S_i)\sum_{j=1}^N W_{ij}A_{ij}\mathbf{H}(\mathbf{x}_j) = (k_i/S_i)\langle W_{ij}\mathbf{H}(\mathbf{x}_j)\rangle_i$ be the weighted local mean field of all the neighbors connected to the oscillator i. Eq. (1) can be rewritten as $\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) + \sigma S_i[\bar{\mathbf{H}}_i^W - \mathbf{H}(\mathbf{x}_i)]$. Since the state $\mathbf{x}_j$ of an oscillator j is not affected directly by the individual output weights W_{ij}, we may assume that W_{ij} and $\mathbf{H}(\mathbf{x}_j)$ are statistically uncorrelated and consequently $\bar{\mathbf{H}}_i^W \approx (k_i/S_i)\langle W_{ij}\rangle_i\langle\mathbf{H}(\mathbf{x}_j)\rangle_i = \bar{\mathbf{H}}_i$ for large k_i. Here $\bar{\mathbf{H}}_i = (1/k_i)\sum_{j=1}^N A_{ij}\mathbf{H}(\mathbf{x}_j)$ is the unweighted local mean field. If the network is sufficiently random with large enough minimum degree $k_{\min}$, the local mean field $\bar{\mathbf{H}}_i$ can be approximated by the global mean field of the network, $\bar{\mathbf{H}}_i \approx \bar{\mathbf{H}}$. Moreover, for small perturbations close to the synchronized state $\mathbf{s}$, we may assume $\bar{\mathbf{H}}_i \approx \mathbf{H}(\mathbf{s})$, and the system is approximated as

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) + \sigma S_i[\mathbf{H}(\mathbf{s}) - \mathbf{H}(\mathbf{x}_i)], \quad (7)$$

indicating that the oscillators are decoupled and forced by a common oscillator $\dot{\mathbf{s}} = \mathbf{F}(\mathbf{s})$, with the forcing strength being proportional to the intensity S_i. If there exists some σ satisfying $\epsilon_1 < \sigma S_i < \epsilon_2$ for all i, then all the oscillators are synchronizable by the common driving $\mathbf{H}(\mathbf{s})$, corresponding to complete synchronization of the whole network. This observation suggests that the eigenratio R can be approximated as

$$R \approx \frac{S_{\max}}{S_{\min}}. \quad (8)$$

The deviation of the the local mean field $\bar{\mathbf{H}}_i$ from the global one $\bar{\mathbf{H}}$ due to finite degree has further affects on the synchronizability. The mean amplitude of the deviation over the whole network is expected to be related to the mean degree K. This effect is similar for different distributions of the intensities. A more detailed dependence on K can be obtained for the special case where the intensities are fully uniform ($S_i = 1\ \forall i$). In this case, the coupling matrix G in Eq. (2) can be written as $G = D^{-1}(I - A)$, where $D = \mathrm{diag}\{k_1, k_2, \ldots, k_N\}$ are the diagonal matrix of the degrees. G has the same spectrum of eigenvalues of the (symmetric) normalized Laplacian matrix $H = D^{-1/2}(I - A)D^{-1/2}$. Recently, it has been shown [20] that the spectrum of H tends to the semicircle law for large random networks with arbitrary expected degrees, provided that the minimum expected degree $k_{\min} \gg \sqrt{K}$, and that $\max\{1 - \lambda_2, \lambda_N - 1\} = [1 + o(1)]\frac{2}{\sqrt{K}}$ for $k_{\min} \gg \sqrt{K}\ln^3 N$. From these, it follows that

$$R \approx R_H(K) = \frac{1 + 2/\sqrt{K}}{1 - 2/\sqrt{K}}, \quad (9)$$

Our simulations on various networks indicate that Eq. (9) already provides a good approximation under the weaker condition $k_{\min} \gg 1$, regardless of the degree distribution.

Based on these physical arguments, we may assume that the contribution due to the number of connections is statistically independent of the contribution due to the strength of the connections, so that we can combine Eq. (8) and (9) to obtain

$$R = A_R \frac{S_{\max}}{S_{\min}} R_H(K), \quad (10)$$

where A_R is a constant of order 1. This formula is expected to be universal for sufficiently random networks with $k_{\min} \gg 1$ and with arbitrary distributions of intensities S_i and degrees k_i (but real spectra of G).

As shown in Figs. 1-3, with a *single* value of parameter $A_R = 0.47$, Eq. (10) approximates the eigenratio R very closely for different networks and weighted coupling schemes (including unweighted networks). This fitting parameter underestimates R slightly only when the intensities become rather homogeneous ($S_{\max}/S_{\min} < 3$). For large K and homogeneous distribution of intensities, we have $A_R = 1$. When the intensities are not uniform, A_R is smaller but still of order 1. Moreover, A_R is almost constant at large values of $S_{\max}/S_{\min}$ and $\Omega/S_{\min}$ for a given K (Fig. 3(b)).

In addition to heterogeneous degrees and weights, many real networks also display high clustering [1] and nontrivial correlation of degrees [21], which are known to have significant influence on percolation transitions [21] and epidemic

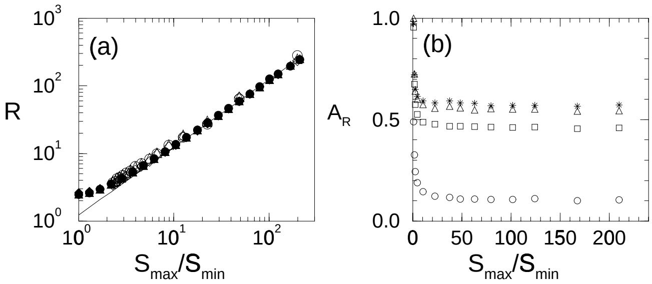

Figure 3: (a) R as a function of $S_{\max}/S_{\min}$. Filled symbols: uniform distribution of $S_i \in [S_{\min}, S_{\max}]$. Open symbols: power-law distribution of S_i, $P(S) \sim S^{-\Gamma}$ for $2.5 \leq \Gamma \leq 10$. Different symbols are for networks with different topologies: growing SFNs with $\alpha = 0$ (circles) and $\alpha = -3$ (diamonds), and K-regular random networks (triangles), for $K = 20$ and $N = 2^{10}$. (K-regular networks are networks where all the nodes have the same degree K.) The solid line is the approximations by Eqs. (10) with $A_R = 0.47$. (b) A_R as a function of $S_{\max}/S_{\min}$. The symbols are for K-regular networks: $K = 5$ ($\circ$), $K = 20$ ($\Box$), $K = 30$ ($\triangle$), and $K = 40$ ($*$). The results are averaged over 100 realizations of networks.

spreadings [22] in complex networks. We find that growing SFNs also exhibits nontrivial clustering and degree correlation for $\alpha < 0$, as shown in Fig. 2(b). In spite of that, the synchronizability is still well accounted by Eqs. (10). This universal formula is thus expected to describe the synchronizability of many realistic networks.

6. Conclusion

In summary, we have shown that the synchronizability of sufficiently random networks with minimum degree $k_{\min} \gg 1$ is universally determined by the mean degree K and the heterogeneity of the intensities S_i. This universality applies to a general class of large networks where the heterogeneity of S_i is due to either the distribution of degrees, as in unweighted SFNs, or the distribution of connection weights, as in weighted K-regular networks, or a combination of both, as expected in most realistic networks. Our universal formula [Eq. (10)] not only describes synchronizability on such weighted complex networks in terms of only two parameters, regardless of the network size, clustering, degree correlation, and details of the degree distribution, but also explains why synchronizability is improved when the heterogeneity of S_i is reduced. This can be useful for network design and control of synchronization.

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