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Criticality in Random Boolean Networks

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Abstract—Boolean networks are considered generic models for a large class of asymmetric networks, such as neural networks or gene regulatory networks. Previous work has shown that such networks undergo a phase transition from an ordered to a 'chaotic' phase as various network parameters (e.g., the connectivity) are changed. Several contributions have established the idea that the network's computational abilities are best near this order-chaos transition. Here, we investigate the stability of the order-chaos transition in the presence of noise and give an upper bound noise level for critically tuned networks that may support an optimal computation. Our studies, however, also reveal shortcomings of the used approaches concerning questions of natural computation.

1. Introduction

The mathematical study of asymmetric networks has been motivated over decades by the interest in biological networks, such as gene regulatory networks [1] or neural networks [2]. For a first approximative description of natural systems, discrete dynamical networks, that are discrete in state and time, are often used. Yet, the analytical treatment of asymmetric discrete networks is challenging due to their non-Hamiltonian character which is a main difference to the symmetric Hopfield networks. A specially simple and generic class of asymmetric networks is defined by Boolean networks which, in a broad sense, are defined by confining the states of the network's elements to the Boolean values $\{0, 1\}$ (or $\{-1, 1\}$). Accordingly, these networks model the interaction of variables that can be either active or inactive. The connectivity of such networks is often chosen as random and sparse. It was argued that this situation likely captures the situation in biological systems [1]. However, recently scale free network structures have been favoured in this respect. As a particular subclass, Random Threshold Networks (RTN) have been investigated in the late eighties as models of diluted asymmetric spin glasses [3]. RTNs show a dynamical phase transition from an ordered to a unordered phase in c, where c is the average connectivity (average number of incoming connections). Typically two discriminating characteristics are used to define the two phases: a) Slightly different initial configurations converge in the ordered phase and diverge in the unordered phase (hence the term 'chaotic' is often used). b) The average length of 'limit cycles' , or also termed: closed orbits (as inevitably reached for systems with a finite configuration space), grows much faster with the system size in the chaotic phase.

In connection with the phase transition from an ordered to a chaotic dynamics the notion of *computation at the edge of chaos* has been developed (e.g., by C. Langton or S.A. Kauffman). According to this idea, extensive computational capabilities are achieved by systems with a dynamics between order and chaos, i.e., at the critical point. Consequently, the idea has been linked more tightly to biology by exploiting real-time computational networks [4] and by addressing questions of self-organised criticality [5]. Most research has been concentrated on studying the role of the c-dependent criticality.A few contributions, on the other hand, addressed the role of noise [6]. While noise-free networks cannot drop out from a once reached limit cycle, noisy networks have the capability to move between different cycles. One limit cycle is typically identified with one possible system behaviour, Thus, according to some ideas ('stochastic resonance'), noise can also enhance the computational abilities of a natural system by facilitating the access to different system behaviours. Here, we first determine the critical connectivity of a rather simple network by means of an annealed approximation. We then address the question whether we can analytically estimate an upper bound of a useful noise level that could support an optimal computation by facilitating change-overs between limit cycles. The notion of computation, however, is rather diffuse or incoherent across the field. E.g., how is optimal computation characterised and what do change-overs mean within the traditional notion of computation? We thus conclude with a general critique and recognise that the further work must inevitably address this issue.

2. Random Threshold Network

We consider networks of randomly interconnected binary elements with states $\sigma_i = \pm 1$, $i = 1, ..., N$. $c \ll N$ is the average connectivity. Furthermore, we restrict ourselves to the simple case of (randomly drawn) binary connection values, i.e., $c_{ij} = \pm 1$, and $c_{ij} = 0$ for no connection. The time evolution is given by the stochastic law (parallel update)

$$\begin{aligned} Prob\,(\sigma_i(t+1) = 1) &= g_\beta(h_i(t)), \\ Prob\,(\sigma_i(t+1) = -1) &= 1 - g_\beta(h_i(t)), \end{aligned} \qquad (1)$$

with

$$h_i(t) = \sum_j c_{ij}\sigma_j(t) + b \qquad (2)$$

and

$$g_\beta(h_i(t)) = \frac{1}{1 + e^{-2\beta h_i(t)}}, \tag{3}$$

where $\beta = 1/T$ controls the (thermal) noise influence and b is a constant bias or threshold.

Thermal noise can also be interpreted as threshold noise. Eq. (1) reduces to a step function for $b \to 0^+$ and $\beta \to \infty$:

$$s\left(h_i(t) = \sum_j c_{ij}\sigma_j(t)\right) = 1 \text{ if } h_i(t) \geq 0, \tag{4}$$

and $s(h_i(t)) = -1$ else.
We now assume a noisy threshold $b_i(t)$ for each node drawn from the distribution $td(b_i) = 2\beta e^{-2\beta b_i}/(1 + e^{-2\beta b_i})$, where β controls the width of the distribution. Thus, the probability that $\sigma_i(t+1) = 1$ is equal to the probability that $h_i(t)+b_i(t) > 0$. This probability is

$$\int_{-\infty}^{h_i(t)} td(b_i)db_i = g_\beta(h_i(t)). \tag{5}$$

2.1. Simulation

To gain a first insight into the phases of the network and their stability in the presence of noise, simulations with networks of size $N = 256$ were performed. As an order parameter, the frozen component $F(c,\beta)$ was found to be useful [5]. $F(c,\beta)$ denotes the fraction of nodes that do not change their state along a limit cycle. If $F \approx 1$, the limit cycles (strictly obtained for $\beta \to \infty$) must be rather short (but not necessarily vice versa), indicating the ordered phase. The results for $b = 0.01$ are shown in Fig. 1. Obviously, $F(c,\beta)$ is decreasing with increasing connectivity c and increasing noise (decreasing β). The transition from the $F \approx 1$ to the $F \approx 0$ phase is, as expected, not sharp for a finite system.

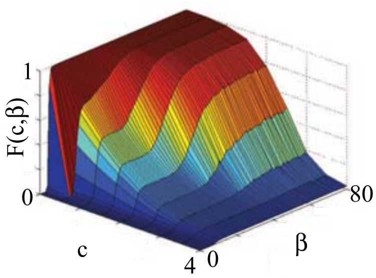

Figure 1: Frozen component.

The transition can be accented by considering an analogue to the magnetic susceptibility defined by

$$\mu(c,\beta) = \langle F(c,\beta)^2\rangle - \langle F(c,\beta)\rangle^2 \tag{6}$$

Fig. 2 reveals a relatively clear transition line (defined by the peaks of (6)) that turns towards smaller connectivities c for higher noise levels ($T = 1/\beta$). However, for high noise levels ($\beta < 30$), the transition line disintegrates and no longer allows to define clear phase boundaries.

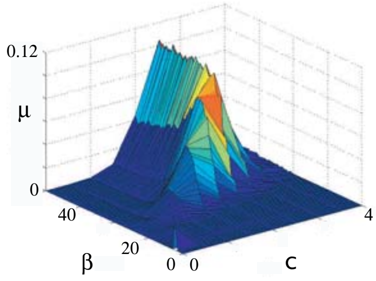

Figure 2: Susceptibility.

2.2. Annealed Approximation: Damage Spreading

The annealed approximation estimates the damage or perturbation spreading in a network under the assumption that the interactions between nodes can be redrawn at each update step. Thus it neglects the fact that the interactions are actually quenched, i.e., constant over time. The approximation finally yields an expression for the time evolution of the normalised Hamming distance $D(t)$ between two configurations σ and σ'. In the noise-free situation ($T = 0$), small perturbations vanish and $D(t) = 0$ is a stable fixed point if the system is in the ordered regime, whereas in the chaotic regime, $D(t) = 0$ becomes unstable and perturbations do not vanish. In the limit $N \to \infty$, D evolves according to

$$D(t+1) = \sum_{k=1}^{\infty} P_k \sum_{p=0}^{k} P_p(D(t)) \cdot f(k,p). \tag{7}$$

P_k is the probability that a node has k incoming connections and is given by a Poisson distribution

$$P_k = \frac{c^k e^{-c}}{k!}. \tag{8}$$

P_p is the probability that p out of these k inputs are different for the configurations σ and σ' with Hamming distance $D(t)$. If we interpret $D(t)$ as the probability that the input of a single node is different for σ and σ', we can write

$$P_p(D(t)) = \binom{k}{p} D(t)^p \, (1 - D(t))^{k-p}. \tag{9}$$

Finally, $f(k,p)$ is the probability that the output of a node differs for σ and σ' given p of k inputs are different. In order to give an expression for $f(k,p)$ for our network (Eq.(1-3)), we write the input fields h_i as

$$h_i(t) = \sum_j c_{ij}\sigma_j(t) + b = x + y + b$$
$$h_i'(t) = \sum_j c_{ij}\sigma_j(t)' + b = x - y + b, \tag{10}$$

where x is the sum over the inputs that are identical and y is the sum over the inputs that are different for σ and

σ'. As x and y are sums of binary variables $-1, 1$ with a priori equal probabilities, the probabilities for x and y are binomial series and $f(k, p)$ evaluates as

$$f(k,p) = \sum_{\substack{x=-k+p \\ \Delta x=2}}^{k-p} \sum_{\substack{y=-p \\ \Delta y=2}}^{p} \binom{k-p}{\frac{(x+k-p)}{2}} \binom{p}{\frac{(y+p)}{2}} \left(\frac{1}{2}\right)^k p_\beta(x,y), \qquad (11)$$

where $p_\beta(x, y)$ is the probability that the output of a node differs for h_i and h'_i given x and y, i.e.,

$$\begin{aligned} p_\beta = & \frac{1}{1+e^{-2\beta(x+y+b)}} \frac{1}{1+e^{2\beta(x-y+b)}} \\ & + \frac{1}{1+e^{2\beta(x+y+b)}} \frac{1}{1+e^{-2\beta(x-y+b)}}. \end{aligned} \qquad (12)$$

2.3. Annealed Approximation: Phase Transition

If $T = 0$, $D = 0$ is a fixed point and the stability analysis

$$d(b) := \frac{\partial D(t+1)}{\partial D(t)}|_{D(t)=0} = 1 \qquad (13)$$

leads to the critical value $c_{crit} = 1.849$ for $b \to 0$. I.e., for $c > c_{crit}$ the fixed point $D = 0$ is instable. The solution of this equation for arbitrary b, using

$$d(b) = \sum_{k>0} \frac{e^{-c}c^k}{(k-1)!} \left(\frac{1}{2}\right)^k \sum_{\substack{x=-k+1 \\ \Delta x=2}}^{k-1} \binom{k-1}{\frac{(x+k-1)}{2}} \left(p_\beta(x,1) + p_\beta(x,-1)\right), \qquad (14)$$

leads to a step-like function of criticality with increasing c_{crit} for larger $|b|$. The steps are due to the integer character of x and y.

For $T > 0$, p_β does not vanish if $h_i = h'_i$ (Eq.12) and thus $D(t + 1) > 0$ even if $D(t) = 0$. This reflects the fact that noise can always lead to a separation of configurations. In fact, for $T \to \infty$, we have $D(t + 1) = 0.5$ irrespective of c, reflecting the randomising effect of T (white area in Fig. 3). This leads to a conceptual difficulty of characterising the phase transition in terms of (13) as $D = 0$ is not a fixed point for any choice of c. Nevertheless, qualitatively the phase transition line as found in Fig. 2 is well reflected by a contour plot showing the levels of $D(\infty)$ (Fig. 3). Such a plot suggests a quasi-critical line starting from $c = 1.849$ for $T = 0$ and going down to $c = 0$

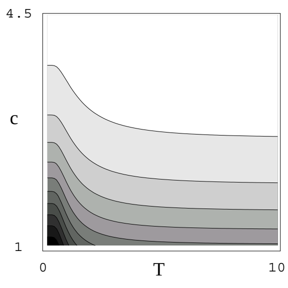

Figure 3: Contour plot for $D(\infty)$.

for increasing noise levels $T > 0$. Our characterisation of the critical line is opposed to the results obtained with $p_\beta = (1+\exp(-\beta(x+|y|+h)))^{-1} - (1+\exp(-\beta(x-|y|+h)))^{-1}$ (as used in [3]). Inherently, this approach desists from separations that are purely noise driven (i.e., $D = 0$ is always a fixed point). It thus allows to determine c_{crit} also for $T > 0$ and yields increasing c_{crit} for higher T. This, however, seems counterintuitive and does not go with our simulations that suggest that c_{crit} rather decreases with T. Yet, the question of an optimal noise level remains open in any case. Inherently, the annealed approach hardly reflects appropriately the complicated landscape of state space regions (here also termed: basins of attractions) whose states lead to the same limit cycles. Ultimately, we have to target the question how c and T influence this landscape. We thus have to slightly shift the focus in the following.

2.4. Antagonistic Role of c and T

In the ordered phase $c < c_{crit}$, limit cycles tend to be longer and perturbations vanish less likely with increasing c. Assuming that the number of limit cycles also grows with c (as found in [1]), the following picture might appropriately sketch the influence of c and T on the landscape structure: Increasing c leads to a more complex attractor landscape. 'Complex' means the occurrence of more (and longer) limit cycles, accompanied by a consecutive segregation of the state space into more basins of attraction. The consequence is twofold. On the one hand, the basins become smaller and the convergence of two different initial states becomes less likely. On the other hand, the average period of a limit cycles grows. Noise, i.e., increasing T in contrary facilitates jumps between different basins of attractions. Consequently, long limit cycles or such with a small basin of attraction hardly show up (completely) during the dynamics as their basin of attraction is left too quickly. Therefore, shorter limit cycles tend to resist the noise whereas longer limit cycles appear to be smoothed out. In summary, c and T play to some extend the role of antagonists: c leads to a more complex attractor landscape, whereas T acts in a smoothing way opposed to an increased complexity. The picture might not be appropriate in the chaotic phase, where the dynamics seems to be dominated by transients.

In the following, we study an equation that shows both, a similar antagonistic role of c and T and a critical connectivity $c_{crit} = 1.849$. It may thus allow to combine the two aspects. The equation describes the probability p_{σ_i} that the output of a node i is $\sigma_i = 1$, having the probability p_j that any input product $c_{ij}\sigma_j = 1$:

$$p_{\sigma_i}(p_j) = \sum_{k=0}^{\infty} \frac{e^{-c}c^k}{k!} \sum_{\substack{x=-k \\ \Delta x=2}}^{k} \binom{k}{\frac{x+k}{2}} p_j^{\frac{k+x}{2}} (1-p_j)^{\frac{k-x}{2}} \frac{1}{1+e^{-2\beta(x+b)}}. \qquad (15)$$

As c_{ij} is randomly chosen from $\{-1, 1\}$, $p_j = 0.5$ is the natural input value. A short calculation confirms that $p_j = 0.5$ is a fixed point of $p_{\sigma_i}(p_j)$ for any β in the case $b \to 0$.

Regarding the function $p_{\sigma_i}(p_j)$, the stability of $p_j = 0.5$ is given by

$$\frac{\partial p_{\sigma_i}}{\partial p_j}|_{p_j=\frac{1}{2}} = \sum_{k=0}^{\infty} \frac{e^{-c}c^k}{k!} \sum_{\substack{x=-k \\ \Delta x=2}}^{k} \binom{k}{\frac{x+k}{2}} \left(\frac{1}{2}\right)^{k-1} \frac{x}{1+e^{-2\beta(x+b)}}. \quad (16)$$

An analysis shows: a) Increasing c destabilises the fixed point. For $T = 1/\beta \to 0$, the critical value ($\frac{\partial p_{\sigma_i}}{\partial p_j}|_{p_j=\frac{1}{2}} = 1$) is just $c_{crit} = 1.849$, i.e., for $c > c_{crit}$ the fixed point is instable and for $c < c_{crit}$ the fixed point is stable. b) Increasing $T = 1/\beta$ leads to a stabilisation. This is clear as $T \to \infty$ inevitably leads to a complete randomisation.

2.5. Virtual Connectivity and Critical Noise Level

The stability analysis of Eq.(15), on the one hand, yields the same critical point as Eq.(13). This can be understood as a small perturbation from $p_j^a = 0.5$ can be modelled by $p_j^b = p_j^a + \Delta p_j$. Obviously, such a perturbation tends to vanish only if $c < c_{crit}$.

On the other hand, the analysis reflects a similar antagonistic role of c and T as in the landscape sketch developed above. As regards (16) an increase in T is as good as a decrease in c. This could inspire the following characterisation of the critical connection for $T > 0$:

$$c_{crit}^T = c_{crit} - \Delta c^T, \quad (17)$$

where $c_{crit} = 1.849$ and Δc^T is the correction term that compensates for the noise T. In order to estimate Δc^T we can directly use that (16) does not distinguish between a decrease in c or an increase in T. Conversely, for each $T > 0$ we can find a $c_T > c_{crit}$ for which $\frac{\partial p_{\sigma_i}}{\partial p_j}|_{p_j=\frac{1}{2}} = 1$. Thus, for a given noise level T, $\Delta c^T = c_T - c_{crit}$ gives us an estimation of the 'cost' in c in order to compensate for the noise.
c_{crit}^T can also be read as the virtual connectivity of a network with $c = c_{crit}$ at noise level T. Consequently, there exists an absolute critical noise level T_{crit} above which no useful computation can be performed anymore (as the networks gets virtually disconnected, i.e., $c_{crit}^T = 0$). A short calculation and Fig. 4 show $T_{crit} = 2.594$. Conversely, for T_{crit} the network can be viewed as being balanced between facilitating change-overs between limit cycles and structure destroying randomness. T_{crit} may thus be proposed as an estimate for a critical or upper bound noise level of a network with connectivity c_{crit}. Yet, more systematic simulations have to prove this hypothesis and the underlying concept.

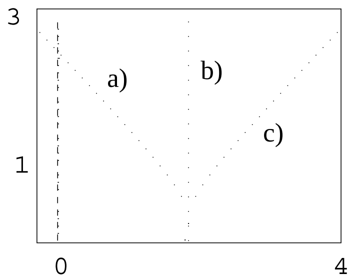

Figure 4: Virtual connectivity: a) c_{crit}^T, b) c_{cit}, c) c_T.

3. Conclusions

Various authors have established the notion of *optimal computation at the edge of chaos*. Following this idea, random threshold networks (RTN) would best perform near the order-chaos transition triggered by the average network connectivity. At this point, the structure of the network's limit cycle (or closed orbit) attractor landscape seems to be most interesting for computational variety or performance. In accordance with the idea that a closed orbit characterises one typical network behaviour, we suggest that noise may play an important role by facilitating change-overs between different behaviours. This raises the question of the best noise level. In this contribution, we gave an estimation for an upper bound or critical noise level for a simple RTN with critical connectivity. The estimation is based on the notion of the antagonistic role of connectivity and noise. The theoretical treatment of RTNs and the related concept of computation, however, remains rudimentary and unsatisfactory. In the future, rigid simulations and a bunch of (novel) concepts might be necessary for our studies, starting with a fundamental review of the notion of 'computation'. Finally, the notions of 'limit cycle' and 'chaos' might also be questioned in order to clarify to what extend concepts of dynamical system theory can be adapted for finite Boolean networks.

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