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Entrainment of driven oscillators and the dynamic behavior of PLL's

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Abstract—Since the so called entrainment phenomenon was observed as disturbance in early radio receivers, several mathematical models were developed by Möller, van der Pol and many others, however, detailed understanding was included in a paper of Andronov and Witt in 1930. At the same time, first ideas were published to use the entrainment effect in a constructive manner for synchronization devices. A related concept for the synchronization of signals was developed in electronics and AFC (automatic frequency control) around 1940 which was denoted as phase–locked loop (PLL) in the late 1950s. Although many papers about the analysis of PLLs were published during the last forty years, no clear relationship between entrainment and PLL behavior is available. In this paper it is shown that the describing equations are very different but they are included in the same class of nonlinear differential equations with limit cycles and an excitation. Furthermore it is shown that both behaviors are essentially the same.

1. Introduction

When first discovered in radio receivers, the phenomenon of pulling and jumping was considered a disturbance [1]. First mathematical models were developed by Moeller [2], van der Pol [4] and Andronov and Witt [5] until in 1935 Rjasin published a detailed description including a transient analysis of the driven van der Pol oscillator [6]. First ideas were proposed using the effect in a constructive manner in the same time in television engineering, e.g. by de Bellescise [7] and Urtel [8]. In 1937 Woodyard published an application for driven oscillators in frequency demodulation [9]. When the effect was further utilized in entrainment circuits by Kaden or Reynauld [10] and the entrainment generator by Urtel [8], there was clearly a change to feedback structures similar to a PLL even though they were still called entrainment circuits. The first block diagrams were published in the early 1950s (e.g. [11] and [12]) and the name PLL was established.

Since both driven oscillators and PLLs are used in a similar manner and have similar functionalities the question arises if there is a connection between these quite different structured devices. Their behavior is inasmuch similar that both are able to lock their phase and frequency to that of an externally provided signal.

2. Fundamentals of driven oscillators and entrainment

The discussion of driven oscillators is restricted to second order oscillators for simplicity reasons, which means that the oscillator is a dynamical system with two state variables possessing a limit cycle in the state space. A very popular equation in the analysis of driven oscillators is the van der Pol equation, which can be derived by an LC element with a parallel nonlinear resistor [5]

$$\ddot{x} + \varepsilon\left(x^2 - 1\right)\dot{x} + x = \Gamma\cos\left(\omega_f t\right) \tag{1}$$

which is a special case of a more generalized equation

$$\ddot{x} + \alpha(x)\,\dot{x} + x = u(t) \tag{2}$$

in which the damping term is a function of the state variable x and the external forcing is a function in time t. The variable x consists of a circuit voltage or current. Therefore, this description level will be called state space from here on. The approach for solving the van der Pol equation is coordinate transformation [13]

$$x(t) = a(t)\cos\left(\omega_f t\right) + b(t)\sin\left(\omega_f t\right). \tag{3}$$

In contrast to the state space, the description in the coordinates a and b is called van der Pol plane. Eq. (3) inserted in Eq. (1) and neglecting $\varepsilon\dot{a}$ and $\varepsilon\dot{b}$, results in a system of differential equations [13]

$$\begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix} = \frac{1}{2}\varepsilon \begin{pmatrix} \left(1 - \frac{1}{4}r^2\right) & -\nu \\ \nu & \left(1 - \frac{1}{4}r^2\right) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \frac{1}{2}\varepsilon \begin{pmatrix} 0 \\ \gamma \end{pmatrix} \tag{4}$$

where ν is detuning, γ the amplitude of the excitation and $r^2 = a^2 + b^2$

$$\nu = \frac{(\omega_f^2 - 1)}{\varepsilon\omega_f}, \quad \gamma = \frac{\Gamma}{\varepsilon\omega_f}. \tag{5}$$

The analysis regarding equilibrium points leads to figure 1 in which equilibrium points are displayed for different parameters γ and ν. It is necessary to distinguish between weak ($\gamma < 1.089$) and strong excitation ($\gamma < 1.089$) because of different bifurcation behaviors which occur at the transition point from entrainment to the loss of entrainment. In the case of weak excitation there are three equilibrium points when entrainment occurs, a stable node, a saddle point and an unstable spiral. At the point of entrainment

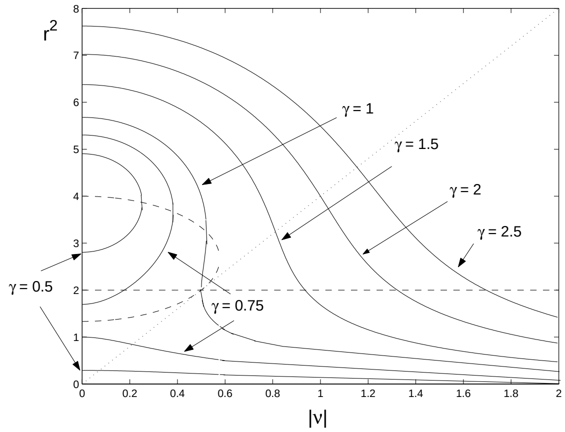

Figure 1: Equilibrium points of the driven van der Pol equation

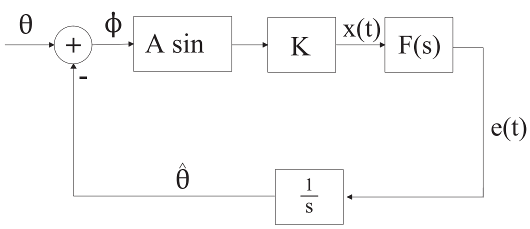

Figure 2: Baseband model of a PLL

loss a stable limit cycle appears via saddle node bifurcation while at strong excitation Andronov-Hopf bifurcation appears when the stable spiral switches to an unstable spiral and a limit cycle [14]. Due to the limit cycle in the van der Pol plane, outside the entrainment region there is a superimposed frequency in the output oscillation. A detailed analysis of the van der Pol equation is presented in a paper by Guckenheimer [15].

Beside the analysis in the van der Pol plane there is a possibility of describing a driven oscillator in the phase plane. The analysis by Fack [16] establishes that the phase difference between the input and oscillator signal φ varies between 0 and $\pm\frac{\pi}{2}$ depending on circuit parameters. The first who derived the equation for the phase of a driven oscillator was Adler [17]

$$\frac{d\varphi}{dt} = -B\sin(\varphi) + \Delta\omega_0 \tag{6}$$

where B is a constant dependent on the amplitudes of the excitation and the internal oscillation and $\Delta\omega_0 = \omega_0 - \omega_f$. Later Pikovski et.al. [14] obtain the same equation with a different approach. In contrast to Adler they use a nonlinear model and pertubation theory. In both cases there is an order reduction of the oscillator model. The internal structure of the oscillator is neglected and the amplitude of the excitation signal only appears as a parameter of the model.

3. PLL fundamentals and derivation of a state space model for the PLL

PLLs are usually described by a block diagram containing a VCO, a phase detector and a low-pass filter [18]. The analysis of the baseband PLL is carried out in the phase space[1] by modelling the components individually as is displayed in figure 2. This model leads to a description based on the following integral differential equation [18]

$$\frac{d\phi}{dt} = \frac{d\theta}{dt} - KA\int_0^t f(t-u)\sin(\phi(u))\,du \tag{7}$$

where $f(t)$ is the pulse response of the filter and K is the open loop gain. The state variables in this description are the phase difference ϕ and its derivatives whilst the amplitude A of the external signal plays no other role than being a parameter. The order of a PLL is determined by the order of the filter in such a way that the order of the PLL results in the order of the filter + 1. In this paper the focus lies on a second order PLL, because the basic behavior of the PLL is not determined by the filter. This is true, since the purpose of using filters of higher order is that the PLL has better tracking behavior in the case of input disturbances. A first order PLL has no filter and the pulse response is $f(t) = \delta(t)$ and Eq. (7) becomes [18]

$$\frac{d\phi}{dt} = \frac{d\theta}{dt} - KA\sin(\phi)\,. \tag{8}$$

For an input phase proportional to time t, $\theta = \Delta\omega t$, this equation is equal to Adlers equation (6). The second order PLL with an imperfect filter can be described by a first order system [18]

$$\frac{d\mathbf{y}}{dt} = \begin{pmatrix} y_2 \\ -\left(\cos(y_1) + \frac{b}{AK}\right)y_2 - \frac{a}{AK}\sin(y_1) + \frac{b}{AK}\omega_\Delta \end{pmatrix} \tag{9}$$
$$\text{with} \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \phi \\ \dot{\phi} \end{pmatrix}$$

in which ω_Δ is the detuning. The equilibrium points $\mathbf{y}_{si}$ $(i = 1, 2)$ of this system result in [18]

$$\mathbf{y}_{s1} = \begin{pmatrix} 2k\pi + \arcsin\left(\frac{b}{a}\omega_\Delta\right) \\ 0 \end{pmatrix} \tag{10}$$

$$\mathbf{y}_{s2} = \begin{pmatrix} (2k-1)\pi - \arcsin\left(\frac{b}{a}\omega_\Delta\right) \\ 0 \end{pmatrix}. \tag{11}$$

When $\omega_\Delta > \frac{a}{b}$ the equilibrium points disappear and the PLL is not able to lock. This means that no constant phase difference ϕ exists and ϕ increases continuously. The behavior of the PLL outside the locking area can be compared to the driven oscillator. The output of the oscillator and the input signal are 90–degree phase–delayed.

[1] In the following, the phase space, van der Pol plane and state space will be referred to as different description levels.

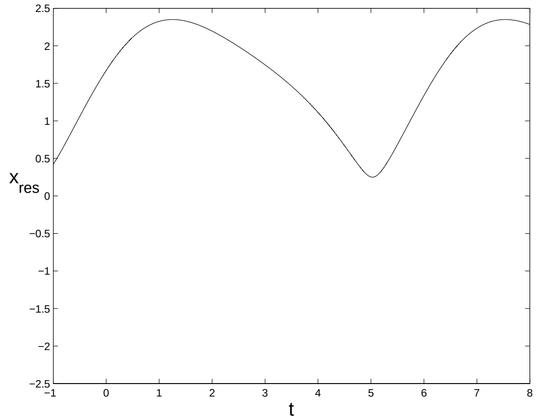

Figure 3: x_{res} of Urtel's model

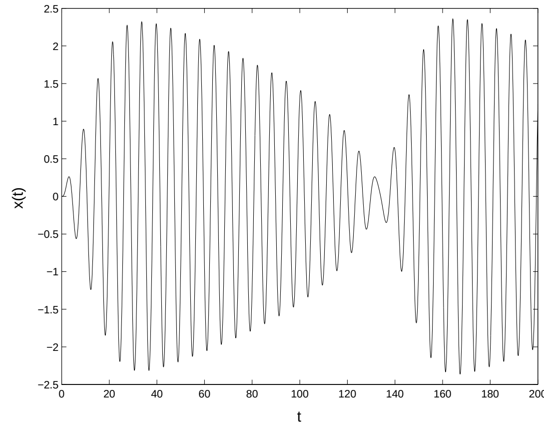

Figure 4: Simulated x_{res} of the driven van der Pol oscillator

Since the baseband model uses a very simple model for the oscillator – an integrator, the model cannot reproduce the behavior of a real PLL in every detail. Shibutani et.al. [19] extended the model for the oscillator to improve the modelling of the transient behavior. Here, a state space model is developed to underline its complex structure. The VCO is modelled by a van der Pol oscillator in which the capacitor C is a function of its input voltage ξ

$$C(\xi)\frac{d^2y}{d\tau^2} + \varepsilon\left(y^2 - 1\right)\frac{dy}{d\tau} + \frac{1}{L}y = 0 \tag{12}$$

Even if the filter is realized with a simple RC–element

$$\frac{d\xi}{d\tau} + \frac{1}{R\hat{C}}\xi = \hat{K}\cdot y\cdot u(t) \tag{13}$$

it seems rather difficult to achieve an approximate solution for this system of nonlinear differential equations with time varying coefficients. Interesting enough it is possible to discuss the dynamical behavior of the phase of the PLL.

4. Relations between the driven oscillator and the PLL

As is demonstrated in the previous sections, it is difficult to compare driven oscillators and PLLs since even their description level is different. In case of the driven oscillator the conventional description level is either the state space or the van der Pol plane. The baseband PLL is usually described in the phase plane even though a state space description is possible, it is very complex due to the feedback structure and the VCO. Comparing the state space description of the PLL, Eq. (12) and Eq. (13), with that of the driven second order oscillator, Eq. (2), it is not possible to establish a simple connection between these.

In 1938, Urtel [8] introduced a tube circuit modelling the entrainment phenomenon which clearly has a core that can be interpreted as a PLL. Beside the PLL core, there is an additional element that combines the input signal and the output signal of the VCO by adding them, see figure 5. This circuit models the behavior of an entrained oscillator where x_{res} represents the state variable of the driven oscillator. Since $u(t)$ and $y(t)$ are 90–degree phase–delayed, the output x_{res} results in

$$x_{res}(t) = K_o\sin\left(\frac{\pi}{2} - \omega_0 t\right) + A\sin\left(\omega_f t\right) \tag{14}$$

This explains the varying phase differences of driven oscillators since the superposition of different weighted $\sin(\omega t)$ and $\cos(\omega t)$ result in $\sin(\omega t + \tilde{\varphi})$. Interpreting $u(t)$ and $y(t)$ as phasors with the frequency ω_f at rest and with

$$\omega_0 = \omega_f + \frac{d\varphi}{dt} \tag{15}$$

the amplitude of $x_{res}(t)$ yields

$$|x_{res}| = \sqrt{K_o^2 + A^2 + 2AK_o\sin(\varphi)}. \tag{16}$$

The same characteristic was derived by Fack [16] in a different way. Outside the entrainment region, the phase φ does not result in a constant phase but a continuously increasing phase. For φ increasing in time the resultant x_{res} is displayed in figure 3 with the approximation $\varphi \approx \frac{1}{2}(2t + \sin(t))$, since φ deviates, due to the dependence of Eq. (15) and $\omega_0 \sim \sin(\varphi)$, from linearly increasing. Comparing it to figure 4 which displays x_{res} of a simulated van der Pol oscillator with similar parameters but a different time scale, it is clear that Eq. (16) describes the envelope of a driven oscillator. The behavior of Urtel's model and the driven van der Pol oscillator is therefore comparable. The additional adder in Urtel's model can be interpreted as an observer of the PLL which transforms the state variables of the PLL into those of the driven oscillator. This simple addition explains the shape of the envelope of the driven oscillator as is shown in these figures.

Regarding the state space models, this means that there is a function F(y,u) which transforms the state space equations, Eq. (12) and Eq. (13), into the equation of the driven oscillator, Eq. (2)

$$x = F(y, u(t)). \tag{17}$$

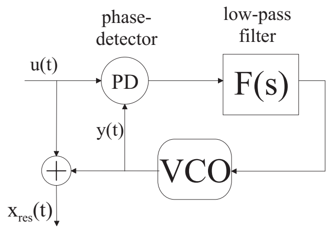

Figure 5: Model of the entrainment generator presented by Urtel

Examining the phase space descriptions this observation is confirmed, since the phase space equations are the same for a driven oscillator (Adler's equation, Eq. (6)) and the first-order PLL, Eq. (8). In microwave theory it has been known that the phase space descriptions of the driven oscillator and PLL are equal [20][2] even though no connection between the dynamical models was established.

This suggests that the PLL was developed as a device that realizes the phase space characteristics of a driven oscillator using separate elements to accomplish this. Therefore the state space models of the PLL and driven oscillator are very different but with the transformation function $x = F(y, u(t))$ their relation can be described.

5. Conclusion

The Adler equation, which is equal to the phase description of the first order PLL, can be derived by pertubation methods from the state space equations of the driven oscillator [14] while the relation between the PLL and the Adler equation is established by modelling the PLL. Since a state space description for a second order PLL was established in this paper the question arises if it is possible to derive the Adler equation from the state space description. Since the baseband PLL is usually described in the phase space, the relation between the driven oscillator and PLL allows a better understanding of the PLL and therefore might lead to an improvement of the modelling of the PLL.

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