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Invariants: A Group Theoretical Foundation and some Applications in Signal Processing and Pattern Recognition

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Abstract—Many signal processing problems can be described by a black-box model. In this paper the input of the black-box is an object, the black-box is a measurement device and the output are the measurements produced. A typical example of an object is a sheet of white paper illuminated by some source. A camera is the measurement device and the produced image is the measurement.

The black-box has a number of internal degrees of freedom that are unrelated to the input but that will affect the produced output. In the example we are interested of the properties of the sheet of paper but the image depends on the illumination and the camera used, the geometrical relation between camera, illumination and paper, the time when the measurements are recorded and many other additional factors. In many applications we want to extract properties of the object from the measurements, independent of the state of the black-box. This is the basic motivation behind all invariance frameworks in pattern recognition and signal processing.

In this paper we will assume that the variations of the measurement device can be modeled by transformation groups. We will then first describe a class of invariants that can be derived from the theory of group representations. We call these invariants integral invariants. Then we introduce another class of invariants derived from the Lie-theory of differential equations. We call them differential invariants.

We will illustrate the general theory with some examples from color constancy, pattern recognition and linear system theory.

1. Introduction

One of the most important properties of natural and technical systems is their ability to adapt to changing environment conditions. For humans these adaptation processes are so basic that we are almost never aware of them. A few of the most well-known examples are:

Size and orientation changes: compensation of the effects of changing the geometrical relation between object and observer

Color constancy describing the adaptation to changing illumination conditions

Shape deformations with face recognition as an important example

Ordering: related to the permutation of objects

The study of these and related perceptional mechanisms has a long tradition in psychology [1] but it has also received a lot of attention in engineering research, for example in computer vision. A recent search (June 2005) on "*invariant* AND computer vision*" in the Science Citation Index resulted in 189 citations (see also [9, 8] for a collection of earlier investigations).

2. Invariants and transformation groups

In the following we will use a black-box model to describe our processing of the input signal. We will assume that all functions are elements in a Hilbert space $\mathbf{H}$ with a scalar product $\langle\ ,\ \rangle$. We assume that our processing unit is defined by a linear operator from the Hilbert space to the real or complex numbers. From the Riesz representation theorem [13] follows that we can find a (filter-) element $f \in \mathbf{H}$ such that the output $o \in \mathbb{C}$ of the unit is given by the scalar product between the input signal $s \in \mathbf{H}$ and the filter

$$o = \langle f, s\rangle$$

Before we can define an invariant we have to specify the operations under which the system is invariant. We will consider operations that modify the signal in some way, i.e. we consider transformations $\mathbf{T} : \mathbf{S} \to \mathbf{S}; s \mapsto \mathbf{T}(s)$. We use the index $g \in G$ to denote these transformations and sometimes we will write $\mathbf{T}_g(s) = s^g = g(s)$. We define that the system is invariant under the transformations in G if

$$o = \langle f, s\rangle = \langle f, s^g\rangle; \qquad \forall g \in G$$

For arbitrary G it is difficult to derive general results regarding the existence and form of invariants and in the following we will always assume that G is a group. Some of the results can be generalized for semi-groups but we will not discuss this case here.

In many cases the signals are defined on some domain $\mathbf{D}$ and the transformations g are mappings: $g: \mathbf{D} \rightarrow \mathbf{D}$. The transformed signals are then given by $s^g(t) = s(g^{-1}(t))$. Typical examples are time-shifts $\mathbf{D} = \mathbb{R}, g(t) = t - \Delta$ or rotations.

3. Integral invariants

The first class of invariant we call integral invariants since they are, essentially, computed by an integration. Since integration often contains a kind of smoothing component we can expect that the invariants obtained in this way are usually insensitive to noise. A typical example are signals $s(\varphi)$ defined on the circle with the rotations with rotation angles ψ as transformations $g: s^g(\varphi) = s(\varphi - \psi)$. We find an easy invariant by averaging the signal over the circle, ie.: $o = \langle 1, s \rangle$. In the following we sketch a generalization of this idea.

We consider the transformations $\mathbf{T}_g$ as mappings $\mathbf{H} \rightarrow \mathbf{H}$ and require $\mathbf{T}_g\mathbf{T}_h = \mathbf{T}_{gh}; \quad \forall g, h \in G$. We will assume that the scalar product in the Hilbert space has the property that all $\mathbf{T}_g$ are unitary operators, ie.: $\langle f, s \rangle = \langle f^g, s^g \rangle \quad \forall g \in G$.

The mapping $g \mapsto \mathbf{T}_g$ is thus a mapping from the group into the space of unitary operators on the signal space. One of the main results of the theory of group representations shows that there is a orthonormal decomposition of $\mathbf{H}$ such that each each component is invariant under all transformations in G:

$$\mathbf{H} = \mathbf{H}_1 \oplus \mathbf{H}_2 \oplus \ldots$$

with $s^g \in \mathbf{H}_k \quad \forall g \in G, s \in \mathbf{H}_k$. We can assume that the spaces $\mathbf{H}_k$ are the smallest possible and then the theory of group representations gives a complete overview over all possible such spaces $\mathbf{H}_k$ or it provides tools to construct them from other, known spaces. The general theory provides also tools to compute the dimensions of the subspaces $\mathbf{H}_k$ without the need to actually construct them.

For a given signal s the above decomposition gives an expansion:

$$s = s_1 + s_2 + \ldots$$

with $\langle s_k, s_l \rangle = \delta_{kl}$ and $s_k, s_k^g \in \mathbf{H}_k, \quad \forall g \in G$. For the invariants we find:

$$\|s\|^2 = \sum_k \|s_k\|^2 = \sum_k \|s_k^g\|^2 = \|s^g\|^2$$

and the norms $\|s_k\|^2$ are all invariants.

We illustrate the construction with some examples.

Finite Groups and Signal Processing Consider a set of points $\mathbf{D} = \{p_k, k = 1, \ldots K\}$ on a two-dimensional grid. We require that the grid points define regular polygons and that the set of points is invariant against all transformations in the symmetry group of these polygons. For the square grid this group is known as dihedral group D(4) and consists of all rotations with angles 0, 90, 180 and 270 degrees and all reflections on the diagonals of the square. (D(6) is the symmetry group of hexagonal grids and has similar properties). The signal space consists of all functions s defined on the domain $\mathbf{D}$ and the dihedral group acts on these signals by a transformation (rotation/reflection) of the points in the domain $\mathbf{D}$. The signal space has dimension K and the theory of group representations shows that the smallest invariant subspaces have dimensions one and two. Typical two-dimensional components are obtained by gradient-type filters in the x- and y-direction. The corresponding invariants are the magnitude of the resulting filter result vectors. They correspond to measures of edge-strength.

Shift-Invariance and Fourier Transform In this case the signals s are functions $s(\varphi)$ on the unit circle and the transformations g are shifts or 2-D rotations: $s^g(\varphi) = s(\varphi - \psi)$. From the algebraic property that all 2-D rotations commute we find that all the smallest, invariant subspaces $\mathbf{H}_k$ have dimension one. In this case the decomposition is given by the subspaces $\mathbf{H}_k = \left\{e^{ik\varphi}, 0 \leq \varphi \leq 2\pi\right\}$ and the signals are written as Fourier series:

$$s(\varphi) = \sum_k c_k e^{ik\varphi}$$

with complex coefficients c_k. The invariants are the absolute values $|c_k|$. For $k = 0$ this gives the averaging invariant mentioned previously.

3-D Rotations and Spherical Harmonics This is similar to the last case but now we replace the circle with the surface of the unit sphere and the 2-D rotations are replaced by their 3-D counterparts. The group is no longer commutative and therefore the invariant subspaces are no longer one-dimensional. In this case it can be shown that the dimensions of the smallest invariant subspaces $\mathbf{H}_k$ are $2k + 1$ and the decomposition is

$$s = \sum_{k=0}^{\infty} \sum_{l=-k}^{k} c_{kl} Y_k^l$$

where Y_k^l are the surface harmonics. The corresponding invariants are: $\left\|(c_{kl})_{l=-k}^k\right\|$.

In many cases the signals g, the operations $\mathbf{T}_g$ of the group and the signal space are naturally given. If the group G is too "large" then the approach described above may result in the case that all invariants are constant. In that case it is possible to construct non-trivial invariants by combining the invariant subspaces with the help of the tensor product and decomposing the resulting product space. The group theoretical structure leads also to fast implementation algorithms and, under certain conditions, to a partial decorrelation of the data.

Early contributions based on this approach can be found in [3, 4, 10, 11]. Application to projection and permutation invariants are found in [7] and recent a description of fast filter systems is given in [5, 6].

4. Differential Invariants

Also in this approach we assume that the transformations form a group but in addition we require that the groups have a differential structure. We consider only matrix groups, ie. groups whose elements are matrices. We will therefore use capital letters M, N to denote the group elements. We also assume that all group elements can be constructed via the matrix exponential:

We assume that there is a Lie-algebra, ie. a vector space $\mathfrak{g}$ with a skew-symmetric bilinear map $[H, J]$ that satisfies the Jacobi identity

$$[[H,J],K]+[[J,K],H]+[[K,H],J]=0$$

and that each element M in the group G has the form $M = \mathrm{e}^J = E + J + J^2/2 + \ldots + J^k/k! + \ldots$ with $J \in \mathfrak{g}, E$ is the identity matrix. We denote elements in the Lie-algebra by capital letters H, J, K. For an element $J \in \mathfrak{g}$ and a real parameter t we define the group elements $M(t) = M_t = \mathrm{e}^{tJ}$. For a function $f(x)$ and a one-parameter transformation group $M(t)$ we can define the derivative:

$$\left.\frac{df(M_t(x))}{dt}\right|_{t=0}$$

This construction shows that every one-parameter group $M(t)$ defines a differential operator. These operators form a vector space of the same structure as the Lie-algebra.

Now assume that the function f is invariant under the group G. It is therefore invariant under all the one-parameter subgroups $M(t)$ of G and it is therefore a solution of the partial differential equation $0 = \frac{df(M_t(x))}{dt}|_{t=0}$. In the case where the corresponding Lie-algebra is a finite-dimensional vector space spanned by the basis elements $J_1, ..., J_L$ it is sufficient to require that the invariants are solutions of the system of L partial differential equations defined by

$$0 = \left.\frac{df(\mathrm{e}^{tJ_l}x)}{dt}\right|_{t=0} \qquad l = 1, \ldots, L$$

We will now illustrate the general approach with an example from color image processing:

In many applications we are interested in the physical properties of objects independent of capturing conditions such as illumination and geometry changes. The interaction between illumination and objects in the scene is very difficult and the topic of ongoing research. Previous studies are, therefore, mostly based on simpler semi-empirical models such as the Dichromatic Reflection and the Kubelka and Munk Model [12, 2]. Here we use the Dichromatic Reflection Model as an example that illustrates how to construct differential invariants.

The Dichromatic Reflection Model [12] describes the relation between the incoming illumination light and the reflected light (measured by the camera) as a mixture of the light reflected from the surface and the light reflected from the material body. The model assumes that the light $L(x, \lambda)$ reflected from an object can be decomposed into two additive components, an interface (specular) and a body (diffuse) reflectance:

$$L(x,\lambda) = m_S(x)R_S(\lambda)E(\lambda) + m_D(x)R_D(\lambda)E(\lambda)$$

Here x denotes a pixel in the image, $R_S(\lambda)$ and $R_D(\lambda)$ are the specular and diffuse reflectance spectra, $E(\lambda)$ is the spectral power distribution of the incident light and the weighting factors $m_S(x), m_D(x)$ contain the information about geometric characteristics like the angle of incidence light, the angle of remittance light and the phase angle. The measured pixel values $C_n(x)$ using N filters with spectral sensitivities given by $f_1(\lambda)...f_N(\lambda)$ are computed by:

$$\begin{aligned} C_n(x) = \int f_n(\lambda)\,[m_S(x)R_S(\lambda)E(\lambda) \\ +m_D(x)R_D(\lambda)E(\lambda)]\,d\lambda \\ = m_S(x)S_n + m_D(x)D_n \end{aligned}$$

Two object points of the same material have identical reflectance functions and they only differ with respect to their geometrical properties. Two neighboring pixels x_1 and x_2 are likely to consist of the same material and for channel n the pixel values are:

$$\begin{bmatrix} C_n(x_1) \\ C_n(x_2) \end{bmatrix} = \begin{bmatrix} m_S(x_1) & m_D(x_1) \\ m_S(x_2) & m_D(x_2) \end{bmatrix} \begin{bmatrix} S_n \\ D_n \end{bmatrix} = M \begin{bmatrix} S_n \\ D_n \end{bmatrix}$$

The effects of the geometrical properties are collected in the matrix M operating on the vectors $(S_n\ D_n)'$. We can now construct invariants under geometrical transformations by constructing invariants for various subgroups of the group of 2×2 matrices. Examples of such subgroups are (1) 2-D Rotations (1 parameter), (2) Uniform scalings (1 parameter), (3) Non-uniform scalings (2 parameters) and (4) Shears (1 parameter).

Lie-theory provide the tools to investigate properties of systems of PDE's with algebraic methods. An example: assume you want to construct invariants for rotations and shears. Then the properties of Lie-algebra requires the inclusion of the scaling operations as well. Instead of invariants for a 2-parameter group it is necessary to consider a three-parameter group.

Our model separates the geometrical properties (described by M) and the spectral properties. For an $N-$channel camera this leads to a model where the matrix operates on collections of N vectors simultaneously (ie. it operates on the space $\left(\mathbb{R}^2\right)^N$):

$$\begin{bmatrix} C(x_1) \\ C(x_2) \end{bmatrix} = M \begin{bmatrix} S \\ D \end{bmatrix}$$

A general result from Lie-theory now states that there are $2N - k$ functionally independent invariants where k is the dimension of the Lie-algebra. For the case of RGB images and the full matrix group we have $2 \cdot 3 - 4 = 2$ invariants. In many cases these invariants (the solutions of the system of PDE's) can be computed by symbolic mathematics software like Maple. For RGB pixels, denoted by $C(x_k) = (r_k, g_k, b_k)$, we find the following two invariants:

$$\mathrm{f} = \mathrm{F}\left(\frac{-g_2\, b_1 + b_2\, g_1}{r_1\, g_2 - r_2\, g_1}, \frac{b_2\, r_1 - r_2\, b_1}{r_1\, g_2 - r_2\, g_1}\right)$$

Starting with rotations and shearing with two variables we saw that we need to include scalings, the generated Lie-algebra has dimension three and there are three independent invariants:

$$\mathrm{f} = \mathrm{F}(r_1\, g_2 - r_2\, g_1,\, b_2\, r_1 - r_2\, b_1, \frac{-g_2\, b_1 + b_2\, g_1}{r_1\, g_2 - r_2\, g_1})$$

Here F is an arbitrary function of its arguments.

5. Conclusions

We showed that group theory provides a number of tools to construct invariants. We divided these methods in two classes: integral and differential invariants. The integral invariants presented here have their roots in the theory of group representations. Differential invariants, on the other hand, are derived with tools from Lie-theory and differential equations. It is impossible to give a comprehensive overview over this vast field in such a short article and we have to refer the reader to the literature.

Finally we want to make two remarks: Invariants have been popular and have been re-discovered many times. The approach here goes at least back to Weyl, one of the founders of modern group theory, and his ideas about symmetries. We should also remark that there are strong similarities with conservations laws in theoretical physics (Noether's theorem). Finally we want to point out that progress in the capabilities of mathematical symbolic software, like Maple and Mathematica, makes it possible to automatically solve many of the systems of partial differential equations defined by symmetry groups. Lie-theory does therefore not only answer the question of how many invariants there are for a given problem but it also provide the tools to construct these invariants automatically.

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