2005 International Symposium on Nonlinear
Theory and its Applications (NOLTA2005)
Bruges, Belgium, October 18-21, 2005

On Brayton and Moser's Missing Stability Theorem

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Abstract—In the early sixties Brayton and Moser proved three theorems concerning the stability of nonlinear electrical circuits. The applicability of each theorem depends on three different conditions on the type of admissible nonlinearities in circuit. Roughly speaking, this means that the theorems apply to either circuits that contain purely linear resistors or conductors—combined with linear or nonlinear inductors and capacitors, or to circuits that contain purely linear inductors and capacitors—combined with linear or nonlinear resistors and conductors. This brief note presents a generalization of Brayton and Moser's stability theorems that also includes the analysis of circuits that contain nonlinear resistors, conductors, inductors and/or capacitors at the same time.

1. Background and Motivation

In the early sixties, J.K. Moser [3] developed a mathematical analysis to study the stability of circuits containing tunnel diodes[1]. His method was based on a certain 'potential function', which was four years later generalized and coined 'mixed-potential' by the same author, together with his companion R.K. Brayton, in [1]. Basically, their theory is based on the observation that the differential equations describing the behavior of a large class of nonlinear RGLC circuits can be written in the form

$$Q(x)\dot{x} = P_x(x), \tag{1}$$

where $x = \mathrm{col}(i_1, \ldots, i_\ell, v_1, \ldots, v_c) \in \mathbb{R}^n$, $n = \ell + c$, represents the currents through the ℓ independent inductors (L) and the voltages across the c independent capacitors (C), respectively. The notation $P_x(x)$ denotes the gradient of the scalar function $P : \mathbb{R}^n \to \mathbb{R}$, i.e., $P_x(x) := \partial P(x)/\partial x$. This function—the mixed-potential—in its present form captures all the necessary information about the topological structure (circuit graph), and the characteristics of the resistive elements contained in the circuit. The function $P(x) = P(i, v)$ has the units of power and is constructed as

$$P(i, v) = A(i) - B(v) + N(i, v), \tag{2}$$

where $A(i)$ and $B(v)$ denote the current potential (content) related with the current-controlled resistors (R) and voltage sources, and the voltage potential (co-content) related with the voltage-controlled resistors[2] (G) and current sources, respectively. The functions $A(i)$ and $B(v)$ are assumed to be twice differentiable. The function $N(i, v)$ is determined by the interconnection of the inductors and capacitors:

$$N(i, v) = \sum_{j=1}^{\ell} \sum_{k=1}^{c} \gamma_{jk} i_j v_k,$$

where γ_{jk} represents the interconnection between i_j and v_k. Furthermore, the $n \times n$ matrix $Q(x) = Q(i, v)$ contains the incremental values of the inductors and capacitors, i.e.,

$$Q(i, v) = \begin{pmatrix} -L(i) & 0 \\ 0 & C(v) \end{pmatrix}, \tag{3}$$

of which each entry is assumed to be a differentiable function.

The main motivation of writing the circuit dynamics in the form (1) is that, by means of the mixed-potential, Brayton and Moser were able to proof several theorems concerning global asymptotic stability. Additionally, these theorems offer criteria for the amount of admissible negative resistance and for the analysis of performance. Two of Brayton and Moser's stability theorems—Theorem 3 and 4 of [1], pp. 19 and 21, respectively (which for ease of reference are added in the Appendix)—give conditions which depend on the interconnection of the circuit as given by the matrix γ (formed by the elements γ_{jk}) but are independent of the nonlinearities in either the R, L and C or the G, L and C elements. Some additional requirements of Theorem 3 and 4 of [1] are that either the Hessian[3] of the current or voltage

[1] It should be mentioned that related ideas where already contained in a paper by Stöhr in the early fifties (see [2] for some historical remarks).

[2] Voltage-controlled resistors are often referred to as conductors.

[3] For $K : \mathbb{R}^n \to \mathbb{R}$, we denote $K_{xx}(x) := \partial^2 K(x)/\partial x^2$.

potentials should be *constant* and *positive definite*, i.e., either $A_{ii}(i) > 0$ or $B_{vv}(v) > 0$. Roughly speaking, this means that either *all* inductors should have some linear series resistance or *all* capacitors should have some linear parallel conductor. On the other hand, the third theorem—Theorem 5 of [1], pp. 22—does not depend on the interconnection matrix γ but gives conditions which depend on the nonlinearities in both R and G. However, as a dual to Theorem 3 and 4, this theorem requires linearity of L and C.

In summary, Table 1 shows the assumptions on the circuit elements regarding the applicability of each of the three theorems. The column marked with '?' represents the 'missing' theorem, i.e., a generalization of the existing three theorems. It is the purpose of this brief note to further generalize the results of [1] and fill in the column marked with '?'. In the following sections a new theorem will be proved which omits all possible linearity requirements on the circuit elements and thus completes Table 1. Like in [1], we will assume that $Q(x)$ is globally invertible.

Table 1: Different assumptions for Brayton and Moser's stability theorems; linear (LIN) and nonlinear (NL). The column marked with '?' represents the 'missing' theorem.

Type	Thm. 3	Thm. 4	Thm. 5	?
R	LIN	NL	NL	NL
G	NL	LIN	NL	NL
L/C	NL	NL	LIN	NL

2. Preliminaries

The underlying idea to proof the stability theorems 3, 4, and 5 of [1], is the search for an nontrivial alternative pair, say $Q^\star(x)$ and $P^\star(x)$, other than $Q(x)$ and $P(x)$ such that (1) can be written as

$$Q^\star(x)\dot{x} = P_x^\star(x), \tag{4}$$

and such that the symmetric part of $Q^\star(x)$ is *negative definite*, i.e., such that

$$Q_s^\star(x) := \frac{1}{2}\Big(Q^\star(x) + Q^{\star T}(x)\Big) < 0, \tag{5}$$

for all x. A necessary and sufficient condition for (4) to describe the same dynamics as (1) is

$$Q^\star(x)Q^{-1}(x)P_x(x) = P_x^\star(x). \tag{6}$$

The key observation is that for any *constant* symmetric matrix $M \in \mathbb{R}^{n\times n}$ and arbitrary constant $\lambda \in \mathbb{R}$, a family of suitable pairs can be represented by

$$Q^\star(x) = P_{xx}(x)MQ(x) + \lambda Q(x), \tag{7}$$

$$P^\star(x) = \frac{1}{2}\langle P_x(x), MP_x(x)\rangle + \lambda P(x), \tag{8}$$

where the notation $\langle\cdot,\cdot\rangle$ denotes the usual inner product, e.g., for any $x, y \in \mathbb{R}^n$, $\langle x, y\rangle := x^T y$. This is easily seen since $P_x^\star(x) = P_{xx}(x)MP_x(x) + \lambda P_x(x)$, and therefore

$$\begin{aligned} &Q^\star(x)Q^{-1}(x)P_x(x) = \\ &\qquad P_{xx}(x)MQ(x)Q^{-1}(x)P_x(x) + Q(x)Q^{-1}(x)P_x(x), \end{aligned}$$

which implies the equality of (6), and thus (1) and (4) coincide.

The requirement that M should be chosen constant is precisely the reason for the several linearity assumptions of the theorems in [1] (see Table 1). Hence, the first step towards a generalization of the theorems is to extend the above procedure to $M = M(x)$.

From previous work [4], we know that a generalization of admissible pairs, $Q^\star(x)$ and $P^\star(x)$, is defined by

$$\begin{aligned} Q^\star(x) = \frac{1}{2}&P_{xx}(x)M(x)Q(x) + \\ &\frac{1}{2}\Big(P_x^T(x)M(x)\Big)_x Q(x) + \lambda Q(x), \end{aligned} \tag{9}$$

$$P^\star(x) = \frac{1}{2}\langle P_x(x), M(x)P_x(x)\rangle + \lambda P(x). \tag{10}$$

The proof of this result is fairly simple by noting that

$$\begin{aligned} P_x^\star(x) = \frac{1}{2}&P_{xx}(x)M(x)P_x(x) \\ &+\frac{1}{2}\Big(P_x^T(x)M(x)\Big)_x P_x(x) + \lambda P_x(x), \end{aligned}$$

which again, by using (6), clearly restores the original description (1). The characterization of the pair (9) and (10) is the key to establish our main result.

3. The Missing Theorem

As discussed before, the original form of Brayton and Moser's fifth theorem does not depend on the circuit interconnection matrix γ, but it imposes the condition that the inductors and capacitors are linear. Using the theory developed in the previous section, the latter restriction can be removed as follows. Let

$$K_1(i,v) := \frac{1}{2}A_{ii}(i) + \frac{1}{2}\Big([A_i(i) + \gamma v]^T L^{-1}(i)\Big)_i L(i),$$

$$K_2(i,v) := \frac{1}{2}B_{vv}(v) + \frac{1}{2}\Big([B_v(v) - \gamma^T i]^T C^{-1}(v)\Big)_v C(v),$$

and let

$$K_j^s(x) := \frac{1}{2}\Big(K_j(x) + K_j^T(x)\Big), \quad j = 1, 2$$

denote their corresponding symmetric parts. Furthermore, let the eigenvalues of a symmetric matrix $S(x) \in \mathbb{R}^{m\times m}$ be denoted by the set $\sigma(x) = \{\sigma_1(x), \ldots, \sigma_m(x)\}$, and let $\mu(S)$ represent the *infimum* of the eigenvalues of $S(x)$ for all x, i.e.,

$$\mu(S) = \inf_{x,l} \{\sigma_l(x)\}, \quad l = 1, \ldots, m.$$

Inspired by [1], we start by selecting $M(x) = M(i, v)$ and λ in (9) and (10) as

$$M(i, v) = \begin{pmatrix} L^{-1}(i) & 0 \\ 0 & C^{-1}(v) \end{pmatrix},$$

and

$$\lambda = \frac{\mu_2 - \mu_1}{2},$$

where $\mu_1 := \mu(\tilde{K}_1^s)$ and $\mu_2 := \mu(\tilde{K}_2^s)$ represent the infima of the eigenvalues of the matrices

$$\tilde{K}_1^s(i, v) := L^{-1/2}(i)K_1^s(i, v)L^{-1/2}(i)$$
$$\tilde{K}_2^s(i, v) := C^{-1/2}(v)K_2^s(i, v)C^{-1/2}(v).$$

Hence, by substituting the latter into (9) and (10), we obtain

$$Q^\star(i, v) = \begin{pmatrix} -K_1(i, v) & \gamma \\ -\gamma^T & -K_2(i, v) \end{pmatrix} + \lambda \begin{pmatrix} -L(i) & 0 \\ 0 & C(v) \end{pmatrix},$$

and

$$P^\star(i, v) = \frac{1}{2}\left\langle P_i(i, v), L^{-1}(i)P_i(i, v)\right\rangle + \frac{1}{2}\left\langle P_v(i, v), C^{-1}(v)P_v(i, v)\right\rangle + \lambda P(i, v), \quad (11)$$

which enables us to proof the following theorem.

Theorem 5*: Under the condition that

$$\mu_1 + \mu_2 \geq \delta, \quad \delta > 0, \quad (12)$$

and $P^\star(x) \to \infty$ as $|x| \to \infty$, where $P^\star(x)$ is given by (11), then all trajectories of (1) tend to the set of equilibrium points as $t \to \infty$.

Proof: The proof follows along the same lines as the proof of Theorem 5 in [1], and basically consists in evaluating the sign of

$$\dot{P}^\star(x) = \langle \dot{x}, P_x^\star(x)\rangle = \langle \dot{x}, Q^\star(x)\dot{x}\rangle,$$

for all x, i.e., we need to show that under condition (12) the symmetric part of $Q^\star(x)$ is negative definite. Hence, by defining $y := L^{1/2}(i)di/dt$ and $z := C^{1/2}(v)dv/dt$, and by using (5), one can write (for sake of brevity we omit the arguments in x)

$$\begin{aligned} -\langle \dot{x}, Q_s^\star \dot{x}\rangle &= \langle y, L^{-1/2}K_1^s L^{-1/2} y\rangle + \langle z, C^{-1/2}K_2^s C^{-1/2} z\rangle \\ &\quad + \lambda\Big(\langle y, y\rangle - \langle z, z\rangle\Big) \\ &\geq (\mu_1 + \lambda)\langle y, y\rangle + (\mu_2 - \lambda)\langle z, z\rangle \\ &\geq \frac{\mu_1 + \mu_2}{2}\Big(\langle y, y\rangle + \langle z, z\rangle\Big) \\ &> 0, \end{aligned}$$

for all $y, z \neq 0$, which under condition (12) shows that $-\langle \dot{x}, Q^\star(x)\dot{x}\rangle$ is positive semi-definite and is equal to zero only if and only if $\dot{x} = 0$. Thus, since $\langle \dot{x}, Q^\star(x)\dot{x}\rangle < 0$, for all $\dot{x} \neq 0$, we conclude that $\dot{P}^\star(x)$ is monotone decreasing except at the equilibria. ■

It is directly noticed that if the inductors and capacitors are constant, i.e., $L(i) = L$ and $C(v) = C$, the matrices $K_1(i, v)$ and $K_2(i, v)$ reduce to $K_1(i, v) = A_{ii}(i)$ and $K_2(i, v) = B_{vv}(v)$, respectively. In that case, Theorem 5* reduces to Theorem 5 in [1] (see also the Appendix). However, in case of nonlinear inductors and capacitors, the difference between Theorem 5* and Theorem 5 of [1] are the additional terms involving the derivatives of $L^{-1}(i)$ and $C^{-1}(v)$. We also observe that in contrast to Theorem 5, the stability condition of Theorem 5* now depends on the graph of the circuit since the interconnection matrix γ now appears in K_1 and K_2, and thus in the criterion (recall the discussion about the differences between the three original theorems in Section 1).

4. Discussion

So far, we have derived a new stability theorem that omits the restrictions imposed by the existing stability theorems originally proposed by Brayton and Moser. Our result is mainly based on the generalization of Theorem 5 in [1]. However, the characterization of the pair (9) and (10) also naturally suggest to generalize Theorem 3 and 4 of [1]. This would mean that we have to select the M-matrix in (9) and (10) either as

$$M(i) = \begin{pmatrix} 2A_{ii}^{-1}(i) & 0 \\ 0 & 0 \end{pmatrix},$$

or

$$M(v) = \begin{pmatrix} 0 & 0 \\ 0 & 2B_{vv}^{-1}(v) \end{pmatrix},$$

respectively.

As discussed in Section 1, invertibility of the matrix $A_{ii}(i)$ (resp. $B_{vv}(v)$) means that at least every inductor (resp. capacitor) should contain a series resistor (resp. parallel conductor) with a strictly convex characteristic (Ohmian) relation. These conditions seem more restrictive than the conditions imposed by Theorem 5⋆. For that reason we will not expose any further details herein, and just mention their existence.

References

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Appendix

The three Brayton-Moser stability theorems [1] are:

Theorem 3: If $A_{ii}(i)$ is constant, symmetric and strictly positive, $B(v) + |\gamma v| \to \infty$ as $|v| \to \infty$, and[4]

$$\left\| L^{\frac{1}{2}}(i) A_{ii}^{-1}(i) \gamma C^{-\frac{1}{2}}(v) \right\| \leq 1 - \delta, \quad \delta > 0,$$

for all (i, v), then all trajectories of (1) tend to the set of equilibrium points as $t \to \infty$.

Theorem 4: If $B_{vv}(v)$ is constant, symmetric and strictly positive, $A(i) + |\gamma^T i| \to \infty$ as $|i| \to \infty$, and

$$\left\| C^{\frac{1}{2}}(v) B_{vv}^{-1}(v) \gamma^T L^{-\frac{1}{2}}(i) \right\| \leq 1 - \delta, \quad \delta > 0,$$

for all (i, v), then all trajectories of (1) tend to the set of equilibrium points as $t \to \infty$.

Theorem 5: Under the condition that L and C are constant, symmetric and strictly positive,

$$\mu\left(L^{-1/2} A_{ii}(i) L^{-1/2}\right) + \mu\left(C^{-1/2} B_{vv}(v) C^{-1/2}\right) \geq \delta, \quad \delta > 0,$$

for all (i, v), and $P^\star(i, v) \to \infty$ as $|i| + |v| \to \infty$, all trajectories of (1) approach the equilibrium solutions as $t \to \infty$.

[4] The notation $||K||$ denotes the spectral norm of a matrix.