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Interconnection by Sharing Variables

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Abstract—In this presentation, a formalism for system interconnection is proposed based on the idea is that systems are interconnected by sharing variables. We apply this to modeling by tearing and zooming.

1. Introduction

Systems, especially engineering systems, usually consist of interconnections of subsystems. This feature is crucial in both modeling and synthesis. The aim of this presentation is to formalize interconnections and to analyze the model structures that emerge.

From an applications point of view the input/output framework is much more restrictive than one is often led to believe. 'Arrows', signal flows, dominate systems theory in engineering. There are many situations, for instance in signal processing where the signal flow graph structure is eminently appropriate. However, the architecture formalized by the signal flow arrows is often viewed as being an essential aspect of describing the interaction of a system with its environment. But, the opposite is actually the case, especially for the description of physical systems and for describing their interconnections. In many situations, signal flow graphs they are unphysical, a figment of the imagination, cumbersome, and unnecessary. Sharing common variables is a much more key idea for system interconnection than input-to-output connection.

2. Behavioral systems

Over the last two decades, a framework for the study of systems has been developed that does not take the input/output structure as its starting point. The 'behavioral approach', as this has been called [7, 4], simply identifies the dynamics of a system with a family of trajectories, called the *behavior*, and develops systems theory (including control [8]) from there.

The behavioral framework views modeling as follows. Assume that we have a phenomenon that we wish to describe mathematically. Nature (that is, the reality that governs this phenomenon) can produce certain events (also called outcomes). The totality of possible events (*before* we have modelled the phenomenon) forms a set $\mathbb{U}$, called the *universum*. A *mathematical model* of the phenomenon restricts the outcomes that are declared possible to a subset $\mathfrak{B}$ of $\mathbb{U}$; $\mathfrak{B}$ is called the *behavior* of the model. We refer to $(\mathbb{U}, \mathfrak{B})$ (or to $\mathfrak{B}$ by itself, since $\mathbb{U}$ usually follows from the context) as a mathematical model. As an example, consider the *ideal gas law*, which poses $PV = kNT$ as the relation between the pressure P, the volume V, the number N of moles, and the temperature T of an ideal gas, with k a universal physical constant. The universum $\mathbb{U}$ is $(\mathbb{R}_+)^4$, and the behavior $\mathfrak{B} = \{(P, V, N, T) \in (\mathbb{R}_+)^4 | PV = kNT\}$.

In the study of dynamical systems we are, more specifically, interested in situations where the events are signals, trajectories, i.e. maps from a set of *independent variables* (time, in the present paper) to a set of *dependent variables* (the values taken on by the signals). In this case the universum is the collection of all maps from the set of independent variables to the set of dependent variables. It is convenient to distinguish these sets explicitly in the notation: $\mathbb{T}$ for the set of independent variables, and $\mathbb{W}$ for the set of dependent variables. $\mathbb{T}$ suggests 'time', the case of interest in the present article. Whence a (dynamical) *system* is defined as a triple

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$$

with $\mathfrak{B}$, *the behavior,* a subset of $\mathbb{W}^{\mathbb{T}}$ ($\mathbb{W}^{\mathbb{T}}$ is the standard mathematical notation for the set of all maps from $\mathbb{T}$ to $\mathbb{W}$). The behavior is the central object in this definition. It formalizes which signals $w : \mathbb{T} \to \mathbb{W}$ are possible, according to the model: those in $\mathfrak{B}$, and which are not: those not in $\mathfrak{B}$. The behavioral framework treats a model for what it is: an exclusion law. Of course, in applications, the behavior $\mathfrak{B}$ must be specified somehow, and it is here that differential equations (and difference equations for discrete-time systems) enter the scene.

In the equations describing systems, very often other variables appear in addition to those whose behavior the model aims at describing. The origin of these auxiliary variables varies from case to case. They may be state variables (as in flows, automata, and input/state/output systems); they may be potentials (as in the well-known expressions for the solutions of

Maxwell's equations); most frequently, they are interconnection variables. It is important to incorporate these variables in our modeling language *ab initio*, and to distinguish clearly between the variables whose behavior the model aims at, and the auxiliary variables introduced in the modeling process. The former are called *manifest* variables, and the latter *latent* variables.

A *mathematical model with latent variables* is defined as a triple $(\mathbb{U}, \mathbb{L}, \mathfrak{B}_{\text{full}})$ with $\mathbb{U}$ the universum of manifest variables, $\mathbb{L}$ the universum of latent variables, and $\mathfrak{B}_{\text{full}} \subseteq \mathbb{U} \times \mathbb{L}$ the *full behavior*. It induces (or *represents*) the *manifest model* $(\mathbb{U}, \mathfrak{B})$, with $\mathfrak{B} = \{w \in \mathbb{U} \mid \text{there exists } \ell \in \mathbb{L} \text{ such that } (w, \ell) \in \mathfrak{B}_{\text{full}}\}$. A (dynamical) *system with latent variables* is defined completely analogously as

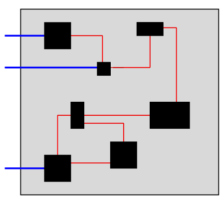
$$\Sigma_{\text{full}} = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_{\text{full}})$$

with $\mathfrak{B}_{\text{full}} \subseteq (\mathbb{W} \times \mathbb{L})^{\mathbb{T}}$. The notion of a system with latent variables is the natural end-point of a modeling process and hence a very natural starting point for the analysis and synthesis of systems. Latent variables enter also very forcefully in representation questions.

The procedure of modeling by *tearing and zooming* is an excellent illustration of the appropriateness of the behavioral approach as the supporting mathematical language. We assume throughout finiteness, i.e., that we interconnect a finite number of modules (subsystems), each with a finite number of terminals, etc. Of course, there are many interconnections that do not fit this 'terminal' paradigm: actions at a distance (as gravity), rolling objects, mixing, components that are interconnected through distributed surfaces, etc.

3. Tearing and zooming

The building blocks, called *modules*, of an interconnected system are systems with *terminals*. Each of these terminals carries variables from a universum, and the (dynamical) laws that govern the module are expressed by a behavior that relates the variables at the various terminals. Finally, the terminals of the modules are assumed to be interconnected, expressed by an *interconnection architecture*. The interconnection architecture imposes interconnection relations between the variables on these terminals.



After interconnection, the architecture leaves some terminals available for interaction with the environment of the overall system. The behavior of the interconnected system consists of the signals that satisfy both the module behavior laws and the interconnection constraints. In specifying the behavior of an interconnected system, we consider the variables on the interconnected terminals as latent variables, and those on the terminals that are left for interaction with the environment as manifest variables. We may think of the interconnected variables *internal* variables, and the exposed variables *external* variables. It is important to note immediately the hierarchical nature of this procedure. The modules thus become subsystems. The paradigmatic example to keep in mind is an electrical circuit. The modules are resistors, capacitors, inductors, transformers, etc. The terminals are the wires attached to the modules and are electrical terminals, each carrying a voltage (the potential) and a current. The interconnection architecture states how the wires are connected. We now formalize all this, assuming that we are treating continuous time dynamical systems (hence, with time set $\mathbb{T} = \mathbb{R}$).

4. Terminals and modules

A *terminal* is specified by its *type*. Giving the type of a terminal identifies the kind of a physical terminal that we are dealing with. The type of terminal implies a universum of *terminal variables*. These variables are physical quantities that characterize the possible 'signal states' on the terminal, it specifies how the module interacts with the environment through this terminal. Some examples of terminals are given in the table below.

Type of terminal	**Variables**	**Universum**
electrical	(voltage, current)	$\mathbb{R} \times \mathbb{R}$
1-D mechanical	(force, position)	$\mathbb{R} \times \mathbb{R}$
2-D mechanical	(position, attitude, force, torque)	$\mathbb{R}^2 \times [0, 2\pi) \times \mathbb{R}^2 \times \mathbb{R}$
thermal	(temperature, heat flow)	$\mathbb{R}_+ \times \mathbb{R}$
fluidic	(pressure, flow)	$\mathbb{R} \times \mathbb{R}$
m-dim. input	$(u_1, u_2, \ldots, u_{\mathtt{m}})$	$\mathbb{R}^{\mathtt{m}}$
p-dim. output	$(y_1, y_2, \ldots, y_{\mathtt{p}})$	$\mathbb{R}^{\mathtt{p}}$
etc.	etc.	etc.

A *module* is specified by its *type*, and its *behavior*. Giving the type of a module identifies the kind of a physical system that we are dealing with. Giving a *behavior specification* of a module implies giving a *representation* and the values of the associated *parameters* a representation. Combined these specify the behavior of the variables on the terminals of the module. The type of a module implies an ordered set of terminals. Since each of the terminals comes equipped with a universum of terminal variables, we thus obtain an ordered set of variables associated with that module. The module behavior then specifies what time trajectories are possible for these variables. Thus a module defines a dynamical system $(\mathbb{R}, \mathbb{W}, \mathfrak{B})$ with $\mathbb{W}$ the

Cartesian product over the terminals of the universa of the terminal variables. However, there are very many ways to specify a behavior (for example, as the solution set of a differential equation, as the image of a differential operator, through a latent variable model, through a transfer function, etc.). The behavioral representation picks out one of these. These representations will then contain unspecified parameters (for example, the coefficients of the differential equation, or the polynomials in a transfer function). Giving the parameter values specifies their numerical values, and completes the specification of the behavior of the signals that are possible on the terminals of a module.

Some examples of modules with their terminals are given below.

Type of module	Terminals	Type of terminals
resistor	(terminal1, terminal2)	(electrical, electrical)
transistor	(collector, emitter, base)	(electrical, idem, idem)
mass, 2 applicators	(appl1, appl2)	(3-D mechanical, idem)
2-inlet vessel	(inlet1, inlet2)	(fluidic, fluidic)
heat exchanger	(inlet, outlet)	(fluidic-thermal, idem)
signal processor	(input, output)	(m-input, p-output)
etc.	etc.	etc.

Some examples of behavioral specifications are given below (only to give an idea of what we have in mind).

Type of module	Specification	Parameter
resistor	default	R in ohms
n-terminal circuit	transfer impedance	$G \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}(\xi)$
n-port circuit	i/s/o admittance	(A, B, C, D)
bar, 2 applicators	Lagrangian equations	mass and length
2-inlet vessel	default	geometry
signal processor	kernel representation	$R \in \mathbb{R}^{\bullet\times\bullet}[\xi]$
signal processor	latent variable	(R, M)
etc.	etc.	etc.

Formally, a system Σ of a given type with T terminals yields $\mathbb{W} = \mathbb{W}_1 \times \mathbb{W}_2 \times \cdots \times \mathbb{W}_T$, with $\mathbb{W}_k$ the universum associated with the k-th terminal. The behavioral specification yields the behavior $\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{R}}$. If $(w_1, w_2, \dots, w_T) \in \mathfrak{B}$, then we think of $w_k \in (\mathbb{W}_k)^{\mathbb{R}}$ as a signal that can be realized on the k-th terminal.

As an example, consider an electrical component. We view this as an device that can interact with its environment through wires. These wires are the terminals. With each terminal we associate two real variables, the potential V and the current I (agreed to be positive when electrical current flows into the device). The laws of the device specify the behavior, which is thus be a subset $\mathfrak{B}$ of $(\mathbb{R}^2 \times \mathbb{R}^2 \times \cdots \times \mathbb{R}^2)^{\mathbb{R}} = ((\mathbb{R}^2)^t)^{\mathbb{R}}$, where t denotes the number of terminal wires. Usually, the behavior $\mathfrak{B}$ will have to satisfy certain restrictions in order for it to qualify as the behavior of an electrical device. For example, *Kirchhoff's current law* and *Kirchhoff's voltage law*. These can be expressed as stating that $((V_1, I_1), \dots, (V_t, I_t)) \in \mathfrak{B}$ must imply $I_1 + I_2 + \cdots + I_t = 0$ and $((V_1 + \alpha, I_1), \dots, (V_t + \alpha, I_t)) \in \mathfrak{B}$ for all $\alpha : \mathbb{R} \to \mathbb{R}$. There may be other requirements, as passivity, etc. The behavior $\mathfrak{B} \subseteq (\mathbb{R}^2 \times \mathbb{R}^2 \times \cdots \times \mathbb{R}^2)^{\mathbb{R}} = ((\mathbb{R}^2)^t)^{\mathbb{R}}$ can be specified in many ways. The circuit could be a three terminal Y or Δ in which case giving the value of 3 resistors is needed to specify the behavior. Or it is a transformer, in which case giving the turns ratio suffices. Or the behavior is a differential equation, with or without latent variables, or it is given in kernel representation, or as a transfer function. We can further think of giving the behavior in terms of port variables, etc.

5. Interconnection architecture

An interconnected system is composed of modules, its building blocks. They serve as subsystems of the overall system. Each module specifies an ordered set of terminals. By listing the modules, and the associated terminals, we obtain the Cartesian product of all the terminals in the interconnected system. The manner in which these terminals, and hence the associated modules, are interconnected is specified by the *interconnection architecture*. This consists of a set of disjunct pairs of terminals, and it is assumed that each such pair consists of terminals of adapted type. Typical 'adapted' type means that they are of the same physical nature: both electrical, or both 1-D mechanical, both thermal, etc. But, when the terminal serves for information processing (inputs to actuators, output of sensors) it could also mean that one variable must be an input to the module to which it is connected (say, the input of an actuator), and the other must be an output to the module to which it is connected (say the output of a sensor).

The interconnection architecture involves only the terminals of the modules and their type, but not the behavior itself. Also, the union of the terminals over the pairs that are part of the interconnection architecture will in general be a strict subset of the union of the terminals of all the modules. We call the terminals that are not involved in the interconnection architecture the *external (or exposed) terminals*. It is along these terminals that the interconnected system can interact with its environment. The terminals that enter in the interconnection architecture are called *internal terminal*. It is along these terminals that the modules are interconnected.

6. Interconnection laws

Pairing of terminals by the interconnection architecture implies an *interconnection law*. Some examples of interconnection laws are shown below.

Pair of terminal	Variables terminal 1	Variables terminal 2	Interconnection constraints
electrical	(V_1, I_1)	(V_2, I_2)	$V_1 = V_2, I_1 + I_2 = 0$
1-D mechanical	(F_1, q_1)	(F_2, q_2)	$F_1 + F_2 = 0, q_1 = q_2$
thermal	(Q_1, T_1)	(Q_2, T_2)	$Q_1 + Q_2 = 0, T_1 = T_2$
fluidic	$(p_1, f_1$	(p_2, f_2)	$p_1 = p_2, f_1 + f_2 = 0$
information processing	m-input u	m-output y	$u = y$
etc.	etc.	etc.	etc.

The physical examples of interconnection laws all involve equating of 'across' variables and putting the sum of 'through' variables to zero. This is in contrast to the input-output identification for information processing terminals. The latter is actually the only interconnection used in flow diagram based modeling, as implemented, for example, in MATLAB's SIMULINK©. It is based on the input/output thinking that permeates systems theory and control. Unfortunately, this is of limited interest for modeling interconnected physical systems. The ideas developed in the *bond-graph* [3] or *port Hamiltonian systems* [5, 6] literature and the modeling packages that use this philosophy are bound to be much more useful in the long run. Interconnection of physical systems involves across and through variables, efforts and flows, extensive and intensive quantities, and not in first instance flow diagrams. These considerations are the main motivation for the development of the behavioral approach.

The resulting graph structure of an interconnected system has the modules in the nodes and the interconnections as the branches. This follows the physics, and should be contrasted with the graph structure pursued in electrical circuit theory, which has the modules in the branches and the elements connections as the nodes. This structures works fine with 2-terminal elements, but is awkward otherwise, and is difficult to generalize to other, non-electrical, domains.

7. Interconnected behavior

We now formalize the interconnected system. The most effective way to proceed is to specify it as a latent variable system, with as manifest variables the variables associated with the external terminals, and as latent variables the internal variables associated with the terminals that are paired by the interconnection architecture. The universum of manifest variables equals $\mathbb{W} = \mathbb{W}_{e_1} \times \cdots \times \mathbb{W}_{e_{|E|}}$, where $E = \{e_1, \ldots, e_{|E|}\}$ is the set of external terminal. The universum of latent variables equals $\mathfrak{L} = \mathbb{W}_{i_1} \times \cdots \times \mathbb{W}_{i_{|I|}}$, where $I = \{i_1, \ldots, i_{|I|}\}$ is the set of internal terminals. Its full behavior behavior consists of the behavior as specified by each of the modules, combined by the interconnection laws obtained by the interconnection architecture. The behavior of each of the modules involves a combination of internal and external variables that are associated with the module. The interconnection law of a pair in the interconnection architecture involves the internal variables associated with these terminals.

A first principles model of an interconnected system always contains latent variables. That is one of the main motivations to introduce latent variables in our modeling language *ab initio*. It also underscores the importance of the *elimimation theorem* [1, 2, 7].

8. Conclusions

Modeling interconnected via the above method of *tearing and zooming* provides the prime example of the usefulness of behaviors and the inadequacy of input/output thinking. Even if our system, after interconnection, allows for a natural input/output representation, it is unlikely that this will be the case of the subsystem and of the interconnection architecture. It is only when considering the more detailed signal flow graph structure of a system that input/output thinking becomes useful. Signal flow graphs are useful building blocks for interpreting information processing systems, but physical systems need a more flexible framework.

9. Acknowledgments

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