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Optimization of Mass and Stiffness Distribution for Efficient Bipedal Walking

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Abstract—Energy-efficient control of bipedal walking robots requires both minimization of mechanical energy losses (often mainly due to impacts) and the use of natural oscillations in a mechanism to minimize actuator torques (as shown by research on passive dynamic walking). In this paper, we discuss how these aspects can be analyzed and optimized using mathematical models of the dynamics, as opposed to using only engineering intuition and experimental results. We use a simple planar three-link robot as an example to illustrate the ideas.

1. Introduction

In research on bipedal walking robots, passive dynamic walking has shown to be a promising approach in applications where efficiency of walking is important. Pure passive dynamic walking, meaning the natural walking motion of an unactuated mechanism on a slope (see for example [1, 2, 3]), is in itself not very useful for applications, since the lack of an active control system makes the walker clearly uncontrollable. Still, it can serve as a starting point for designing mechanisms and controllers that improve stability, robustness, and controllability, but that still rely on the natural walking behavior of the mechanism itself, and hence (hopefully) require little energy. Furthermore, several approaches (such as [4]) explicitly use the knowledge obtained in passive dynamic walking for control.

Instead of approaching the problem of efficient walking by adaptation of pure passive walking down a slope, we can also start from (traditional) actuated walking and use passivity and energy-efficiency ideas to optimize the walking motions. When dealing with actuators, energy efficiency has two aspects: first, the walking gait of the mechanism should be such that little mechanical energy is lost (which is due to friction and the collisions of the feet with the ground). But secondly, practical actuators for robots cannot absorb energy and hence will consume battery power both when doing positive and negative mechanical work.

We propose the following strategy, illustrated in Figure 1, to accomplish this goal. When walking with a certain gait using a traditional controller, energy will oscillate between the walking mechanism and the controller, as in Figure 1a. Since the (practical) actuators in the controller cannot store energy, this energy is dissipated and hence needs to be injected from a battery, leading to in-efficient motion. Instead, looking at Figure 1b, we can use passive mechanical elements (such as springs) to function as temporal storage elements during walking cycles, and let the controller only act at a higher level of control, to increase stability and robustness of the walking motion.

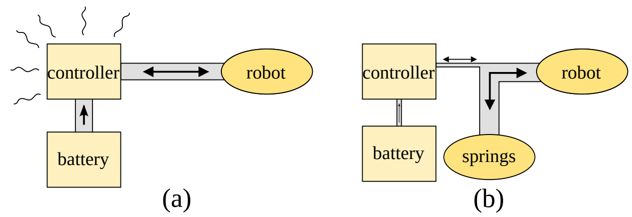

Figure 1: General idea of the presented approach: instead of using a controller (a) that exchanges energy with the robot and dissipates most of it, use passive mechanical elements such as springs (b) for the main energy exchange and the controller only for robustness and higher-level control.

Gomes & Ruina [5] already showed that adding springs can result in walking motions with (ideally) zero energy cost. The approach we present here, though, is more general: we do not search for trajectories for a given spring, but use the springs as degrees of freedom in the search for efficient trajectories. This way, the need disappears to have *a priori* intuitive knowledge of where to place what springs.

We illustrate this approach on a simple planar straight-legged robot with a trunk (upper body), described in Section 2. In Section 3, we analyze the effect of changing the mass distribution and trunk posture on the energy loss due to impact, and we use a numerical optimization routine to find suitable springs to minimize actuator torques. Finally, we discuss the resulting walking motions and describe several possible extensions for future research.

2. Model setup

We consider in this paper the example system of Figure 2: a robot consisting of two unit-length legs and a unit-length trunk, with coordinates q as indicated. The dynamics of this mechanism can be modeled by the Euler-Lagrange equations for mechanical systems with both actuation and constraint forces:

$$\begin{aligned} M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \partial_q V(q) &= \tau + A(q)\lambda \\ A^T(q)\dot{q} &= 0 \end{aligned} \quad (1)$$

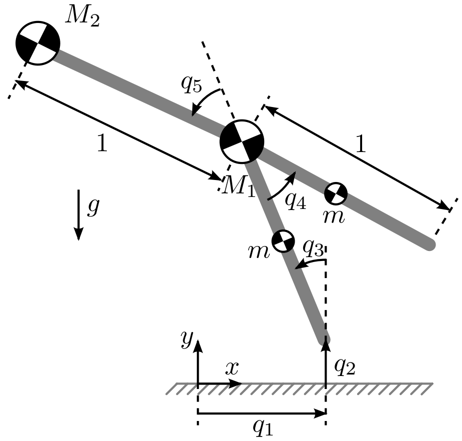

Figure 2: A three-link mechanical walker and the definitions of the coordinates and parameters.

in which $M(q)$ is the mass matrix, $C(q, \dot{q})$ describes Coriolis and centrifugal effects, $V(q)$ is the total potential energy with $\partial_q V$ its partial derivative with respect to q, and τ is the vector of actuation and external torques. Furthermore, $A^T(q)\dot{q}$ defines the vector of constrained velocities of the system, and λ are the collocated constraint forces.

The constraint forces in this model are the ground contact forces on both feet. When only the rear foot of Figure 2 is in contact with the ground, we have $q_2 = \dot{q}_1 = \dot{q}_2 = 0$ and hence the model can be reduced to only three degrees of freedom (q_3, q_4, q_5). Then, when the front leg touches the ground ($q_4 = -2q_3$), the horizontal and vertical velocity of the front foot point are instantaneously set to zero by impulse forces λ acting on the front foot in the horizontal and vertical direction. This can be described by the matrix

$$A^T(q) = \begin{bmatrix} 1 & 0 & 0 & \cos(q_3) & 0 \\ 0 & 1 & -2\sin(q_3) & -\sin(q_3) & 0 \end{bmatrix}$$

On impact of the front leg, the constraint forces λ are such that the velocities after impact satisfy the constraints on the front leg, *i.e.* the front foot velocity is instantaneously zero. We can compute the magnitude of the impulsive constraint forces by integrating the dynamics (1) over the impact, and assuming that the Coriolis and centrifugal effects and all forces except λ are finite during the impact. This results eventually in the expression

$$\dot{q}(t_+) = \left(I - M^{-1}A(A^T M^{-1} A)^{-1} A^T\right) \dot{q}(t_-) \qquad (2)$$

which relates the velocity $\dot{q}(t_-)$ just before impact to the velocity $\dot{q}(t_+)$ just after impact. Equation (2) describes a *projection* of the velocity along the columns of $M^{-1}(q)A(q)$ onto the kernel of the matrix $A^T(q)$. Note that the full five-dimensional mass-matrix is necessary to obtain this projection operation; we cannot just use the reduced model equations in coordinates (q_3, q_4, q_5).

The projection operation (2) not only affects the velocity of the swing foot, but also the velocity $(\dot{q}_1, \dot{q}_2)$ of the stance foot. It can be shown that for most practical situations, the vertical velocity $\dot{q}_2$ of the stance foot instantaneously becomes positive (away from the ground), which means that the stance foot instantaneously becomes the swing foot, and hence the double support phase of the walking motion is instantaneous. Moreover, if we are only interested in walking gaits that are symmetric for the left and right leg, we can relabel the joints immediately after impact, so that q_1 and q_2 now describe the front foot, and q_3, q_4, and q_5 the appropriate angles such that the same configuration as in Figure 2 is obtained, but now with the legs switched.

3. Optimization of the Mechanical Structure to Minimize Actuator Torques

Using the model discussed in the previous section, we now investigate how the mechanical structure of the robot affects its walking behavior. More precisely, we discuss the influence of the mass distribution and posture of the trunk on the energy losses during impact, as well as the use of mechanical springs to minimize actuator torques.

For both aspects, we choose the step length of the walking motion as $2\sin(\frac{1}{6})$ (which also fixes the initial conditions for q_3 and q_4), leg masses as $m = 1$ kg, and the total mass of the trunk as $M_1 + M_2 = 5$ kg. The distribution between M_1 and M_2 as well as the step time T remain variable. We make these choices to keep the example simple and the equations manageable. The ideas of the following section can still be used when the step size is not fixed, and for example only a desired forward speed (ratio of step length and step time) is given.

3.1. Change of Mass Distribution and Trunk Posture

From (2), we can find an expression for the loss of energy during impact, namely as

$$\begin{aligned} \Delta U_k &:= \frac{1}{2}\dot{q}^T(t_+)M(q)\dot{q}(t_+) - \frac{1}{2}\dot{q}^T(t_-)M(q)\dot{q}(t_-) \\ &= -\frac{1}{2}\dot{q}^T(t_-)\left(A(A^T M^{-1}A)^{-1}A^T\right)\dot{q}(t_-) \qquad (3) \end{aligned}$$

This energy loss is hence a quadratic function in the velocity $\dot{q}$ and depends on the posture of the mechanism and its mass distribution.

The matrix $A(A^T M^{-1}A)^{-1}A^T$ is a symmetric, positive semi-definite quadratic form on $\dot{q}$, and hence it can be characterized by a singular value decomposition to find the velocity directions of minimum and maximum energy loss. This standard singular value decomposition would then describe the deformation of the unit sphere $\dot{q}^T\dot{q} = 1$ into an ellipsoid under the operation of the quadratic form. However, the sphere $\dot{q}^T\dot{q} = 1$ has no physical meaning and gives coordinate dependent results, so a much better and physically intuitive decomposition is to study the effect of the quadratic form on the velocities $\dot{q}$ satisfying $\dot{q}^T M\dot{q} = 1$ (a sphere in the metric M), *i.e.* the velocities corresponding to constant kinetic energy. This decomposition can be

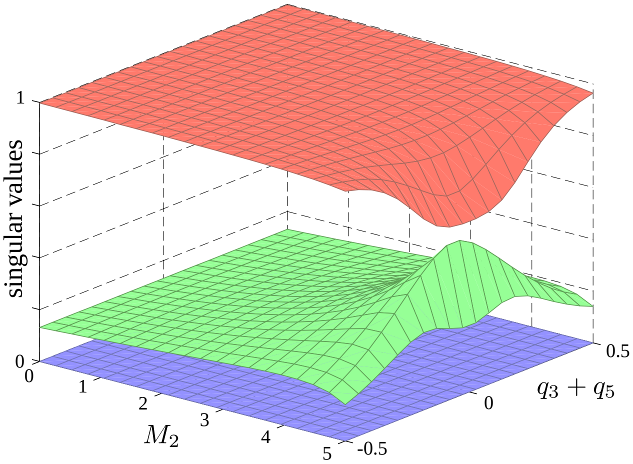

Figure 3: Singular values describing the energy loss on impact for varying postures q_5 and mass ratios $M_1 : M_2$.

written as

$$A(A^T M^{-1} A)^{-1} A^T =$$
$$G^T \left(G^{-T} A(A^T M^{-1} A)^{-1} A^T G^{-1}\right) G = G^T U \Sigma U^T G$$

where $G(q)$ is the Cholesky decomposition of $M(q)$ (such that $M = G^T G$), Σ is the diagonal matrix of singular values, and U is an orthogonal matrix. The singular values Σ describe the effect of the quadratic form $U \Sigma U^T$ on unit-vectors x, and if we parameterize these vectors as $x = G\dot{q}$, then the same singular values describe the effect of the matrix $A(A^T M^{-1} A) A^T$ on velocities $\dot{q}$ satisfying

$$1 = x^T x = (G\dot{q})^T (G\dot{q}) = \dot{q}^T M \dot{q}$$

as desired. The columns of $U^T G$ describe the principle directions $\dot{q}$ corresponding to these singular values, *i.e.* the velocities resulting in minimum and maximum energy loss on impact.

By studying the effect of varying posture and mass distribution on the singular values, we can find what would be the optimal choice for these parameters to minimize mechanical energy loss. Clearly, the real energy loss depends mainly on the direction and magnitude of the velocity $\dot{q}$.

Figure 3 shows a plot of the singular values Σ for various mass distributions and trunk angles. It shows that all singular values are between zero and one, which is clear intuitively, since at most all energy and at least zero energy is lost on impact. Furthermore, the smallest singular value is zero for all parameters, and the corresponding column of $U^T G$ is $\begin{bmatrix} 0 & 0 & * \end{bmatrix}^T$, *i.e.* only a velocity $\dot{q}_5$ of the trunk. This is also clear intuitively, since if the front foot strikes the ground with zero velocity, indeed the contact forces and the resulting energy loss are equal to zero. Finally, the figure shows that the effect of mass distribution and posture is ambiguous: variations of the parameters result in the increase of one singular value and the decrease of another.

3.2. Optimization Using Mechanical Springs

As described in Section 1, efficiency of the walking motion can be increased by using mechanical springs (in addition, *i.e.* in parallel, to the actuators) that provide reversible storage of energy during the normal walking cycle[1]. We now describe a way to find the optimal joint motion and the optimal parameters for these springs, *i.e.* those that minimize the actuator torque requirements during a step. We describe the cost associated with these torques as the approximation of $\int_0^T \tau^2 dt$ by a Riemann sum, *i.e.* as

$$J = \frac{T}{N} \sum_{i=1}^{N} \tau^T(t_i)\tau(t_i) \qquad (4)$$

for some large enough N (we chose $N = 50$). Other cost functions can be chosen instead, for example to penalize ankle torque or peak torque requirements.

We set up the optimization problem as follows. First, we parameterize the joint trajectories of q_3, q_4, and q_5 as fixed-order polynomial functions of time from the beginning to the end of a step. Second, we parameterize the set of allowed springs. For simplicity, we choose here to take two linear springs (with unknown stiffness k_1 and rest length x_0) between the two legs and the trunk, as well as one linear spring between the two legs (with unknown stiffness k_2 and zero rest length for symmetry reasons). The parameters k_1, k_2, x_0 of the springs, together with the step time and the coefficients of the polynomials describing the joint trajectories, are the degrees of freedom in the optimization.

Thirdly, we set up the constraints that the solution of the optimization problem must satisfy. These constraints ensure that the springs have positive stiffness, that the trunk remains close to upright (it does not swing downward between the legs), and that the positions and velocities at the end of a step (after impact and relabeling) are the same as the positions and velocities at the beginning of a step. No explicit initial and final velocities are enforced, only the compatibility relation from the end of a step to the beginning of the next step.

The resulting problem of minimizing (4) is then solved using the Sequential Quadratic Programming (SQP) algorithm implemented in Matlab [6]. The initial search point is simply chosen as $q_3(t) \equiv q_4(t) \equiv q_5(t) \equiv 0$, $k_1 = k_2 = x_0 = 0$, $M_2 = 5$, and $T = 1$. For this example, a (local) optimum was found using a tenth-order polynomial. Figure 4 shows the resulting walking motion, and Table 1 gives the corresponding spring and mass parameters.

4. Discussion and Conclusions

The simulation and optimization show that the use of the springs has resulted in small actuator torques; running the

[1]This idea requires the actuators to be back-driveable, such that they can be freely moved in parallel to the springs without generating a torque.

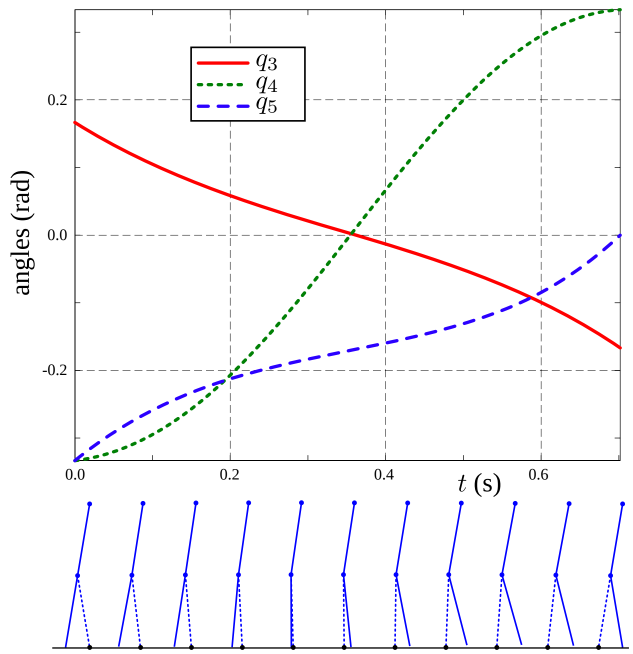

Figure 4: Optimal walking motion minimizing actuator torques when the springs and masses are as in Table 1.

Table 1: Optimal structural parameters minimizing actuator torque, resulting in the motion of Figure 4.

symbol	description	value	unit
T	step time	0.702	s
M_1	hip mass	4.71	kg
M_2	head mass	0.29	kg
k_1	trunk spring stiffness	0.0035	Nm/rad
k_2	leg spring stiffness	0.234	Nm/rad
x_0	trunk spring rest length	5.34	rad

same optimization scheme while keeping the springs equal to zero resulted in a roughly double cost J as well as a larger step time. The fact that still some actuator torque remains (while in [5] a torque-free walking cycle was obtained) is mainly due to the chosen initial estimate (all joints equal to zero) for the optimization routine. The trajectory in [5] contains a reasonably high-frequency, high-amplitude oscillation of the trunk, which is not very close to the chosen initial estimate.

The optimized trunk position and mass distribution of Figure 4 and Table 1 show that the trunk leans slightly forward, and almost all mass is positioned at the hip. This results in a 13% kinetic energy loss on impact, which is slightly below the middle singular value of Figure 3. The optimal parameters for the springs suggest to include a constant torque spring between the trunk and the legs, as well as a linear spring between the legs.

The example shows that the optimization procedure results in natural looking motions without the need for *a priori* intuitive knowledge about suitable initial conditions and spring parameters. Although intuition can clearly help to reduce the number of parameters in the optimization, numerical techniques can help in fine-tuning the remaining parameters.

The presented method can be generalized, for example to include knees, arms, and three-dimensional (as opposed to planar) motion. When using knees with kneecaps, polynomial functions may not be able to capture the stiff dynamics of the knee impact, and hence it may be useful to use piecewise polynomial functions instead (with a non-smooth connection point at the time of knee impact).

As in all non-convex optimization problems with non-linear constraints, the algorithm may find locally optimal solutions, or, especially for higher-dimensional problems, no feasible solution at all. To prevent this, the optimal trajectories obtained for simple robots could be used as an initial guess for more complex robots. For example, the trajectories for the robot in this paper can be used as initial estimate for a planar robot with knees and arms (where the joints for arms and knees could be initialized to zero).

Finally, we want to study variable speed walking, which will most likely result in adjustable physical springs, with different (nonlinear) stiffness for different walking speeds.

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