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Partial entrainment of oscillators in the finite Kuramoto model

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Abstract—We consider the finite version of the Kuramoto model. We prove that (for appropriate — but generic — parameter values) different 'regimes' of partial entrainment can occur.

1. Introduction

The Kuramoto model [2] was introduced to investigate synchronisation in systems of coupled oscillators. A variety of examples of such systems is described in [4]. The model consists of the following differential equations for the phases (θ_i) of the oscillators:

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \qquad \forall i \in \{1, \ldots, N\}, \qquad (1)$$

where N is the number of oscillators in the system, $K \geq 0$ is the coupling strength and the ω_i are drawn from a distribution g. The parameters ω_i represent the individual frequencies of the oscillators that will determine the behaviour of the system for $K = 0$. By picturing the oscillators moving around a circle (e.g. by plotting $(\cos\theta_i, \sin\theta_i)$ in a plane) it can be seen that the interaction between different oscillators is attractive. The interaction will cause the average frequencies of the oscillators to shift away from their individual frequencies towards those of the other oscillators. Kuramoto considered the limit $N \to \infty$ and showed that, if g is unimodal and symmetric about some frequency Ω, there is a critical value K_c of the coupling strength above which a solution exists exhibiting partial synchronisation [3]. For $K > K_c$ this solution is characterised by two different groups of oscillators; those in the first group are locked at the fixed frequency Ω while the other oscillators are drifting around the circle with (average) frequencies different from Ω. The stability properties of this solution are not fully understood yet. In the next section we will give a more detailed description of the solution.

The assumptions Kuramoto made about the distribution do not help anymore when trying to analyse the model for finite N. Analytical results are hard to obtain and mostly refer to special cases such as identical individual frequencies [5] or the case of full synchronisation [1]. In this paper we will address a more general case, although for explicit analytic results we will still need to impose some conditions on the parameters. We will prove that, under these conditions, different 'regimes' of partial entrainment can occur, which can be regarded as an extension of a result in [1].

In the next section we will review the solution proposed by Kuramoto of the model with infinite N as it is described in [3] and we briefly describe an example which we relate to the finite N model. Before investigating general entrainment, we consider a system where two frequencies coincide and state a local stability result for small values of K. In section 4 we will find a lower bound for the coupling strength for the entrainment of a given subset of the population of oscillators. This will allow us to find sufficient conditions for the ω_i that guarantee the existence of several regimes of partial entrainment for different intervals of the coupling strength.

2. Kuramoto's solution

We will consider a slightly more general setting than Kuramoto did by not restricting the distribution g of the individual frequencies to be symmetric. We will look for a solution of (1), for $N \to \infty$, for which there is a group of oscillators, all moving at the same constant velocity Ω_0 while the other oscillators are just moving around the circle with an average velocity different from Ω_0.

The model with infinite N is best described in terms of a population density $\rho(\theta, \omega, t)$ where the values for θ have to be considered modulo 2π. The fraction of the population of oscillators with an individual frequency equal to ω and a phase value between θ and $\theta + \mathrm{d}\theta$ (modulo 2π) at time t equals $\rho(\theta, \omega, t)\mathrm{d}\theta$ for infinitesimally small values of $\mathrm{d}\theta$. This implies that $\int_{-\pi}^{\pi} \mathrm{d}\theta\rho(\theta, \omega, t) = 1, \forall\omega, t$.

The substitution $\theta_i = \theta_i' - \Omega_0 t$ in (1) leads to differential equations of the same form and thus we can simply put $\Omega_0 = 0$, while replacing g by $\tilde{g}$ with $\tilde{g}(\omega) = g(\Omega_0 + \omega)$, without changing the essential dynamics of the system. The synchronised group now moves at zero velocity, which allows us to look for a stationary solution.

The system (1) can be rewritten as

$$\dot{\theta}_i = \omega_i + Kr\sin(\psi - \theta_i), \qquad \forall i \in \{1, \ldots, N\}, \tag{2}$$

by defining r and ψ by

$$re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}. \tag{3}$$

The parameter r can be seen as some kind of order parameter, since when all oscillators are close together, r will be large and when they are spread uniformly over the interval $[-\pi, \pi)$ r will be zero. Note that in general r and ψ will be time-dependent, but since we will look for a stationary solution we assume them to be constant. The substitution $\theta_i = \theta'_i + \psi$ in (1) again doesn't really change the dynamics of the system and thus we can put $\psi = 0$.

If we let $v(\theta, \omega, t)$ represent the velocity of an oscillator with individual frequency ω and phase θ at time t (corresponding with $\dot{\theta}_i$ in (1)) then the evolution of ρ is determined by the continuity equation

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho v)}{\partial \theta}. \tag{4}$$

For a stationary density ρ it follows that whenever $v(\theta, \omega) = v(\theta, \omega, t)$ is nowhere zero for a fixed ω and varying θ, then

$$\rho(\theta, \omega, t) = \rho(\theta, \omega) = \frac{C(\omega)}{v(\theta, \omega)}, \tag{5}$$

where C is some function only depending on ω that will be determined by the normalisation $\int_{-\pi}^{\pi} \mathrm{d}\theta \rho(\theta, \omega, t) = 1$. The velocity v is given by

$$v(\theta, \omega) = \omega - Kr\sin\theta, \tag{6}$$

with

$$r = \int_{-\infty}^{+\infty} \mathrm{d}\omega \tilde{g}(\omega) \int_{-\pi}^{\pi} \mathrm{d}\theta \rho(\theta, \omega)\cos\theta, \tag{7}$$

and thus ρ can be calculated as

$$\rho(\theta, \omega) = \frac{1}{2\pi}\frac{\sqrt{\omega^2 - (Kr)^2}}{|\omega - Kr\sin\theta|}, \tag{8}$$

for $|\omega| > Kr$. For $|\omega| \leq Kr$ the stationarity requires that $\sin\theta = \frac{\omega}{Kr}$ of which we take the attractive solution satisfying $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

$$\rho(\theta, \omega) = \delta\left(\theta - \arcsin\left(\frac{\omega}{Kr}\right)\right), \tag{9}$$

for $|\omega| \leq Kr$. When inserting the value for ρ in equation (7) for r the contribution of the part where $|\omega| > Kr$ cancels out because of the antisymmetry of the integrand under the substitution $\theta \leftrightarrow \pi - \theta$, and thus

$$r = \int_{-Kr}^{Kr} \mathrm{d}\omega \tilde{g}(\omega)\cos\left(\arcsin\left(\frac{\omega}{Kr}\right)\right), \tag{10}$$

or, with $\omega = Kr\sin\theta$,

$$r = Kr\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\theta \tilde{g}(Kr\sin\theta)\cos^2\theta. \tag{11}$$

The solution $r = 0$ ($\rho = \frac{1}{2\pi}$) corresponds with total incoherence (every oscillator moves at its individual frequency). There will be another solution for r to (11) when K is large enough, and since

$$1 = K\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\theta \tilde{g}(Kr\sin\theta)\cos^2\theta \leq K\frac{\pi}{2}\max\tilde{g} \tag{12}$$

we need at least $K \geq \frac{2}{\pi \max \tilde{g}}$ for this. To ensure that these values for r correspond to stationary densities, we still need to impose the continuum version of (3) (with $\psi = 0$). The real part is fulfilled by (7), the imaginary part will determine the value of Ω_0 and is equivalent to

$$\int_{-\infty}^{+\infty} \mathrm{d}\omega g(\Omega_0 + \omega)\int_{-\pi}^{\pi} \mathrm{d}\theta \rho(\theta, \omega)\sin\theta = 0. \tag{13}$$

If g is symmetric about Ω then for $\Omega_0 = \Omega$ this equation is automatically fulfilled because of the antisymmetry of the integrand under the substitution $(\omega, \theta) \leftrightarrow (-\omega, -\theta)$.

In general we can expect the following to happen. For small values of K the incoherent state prevails and every oscillator moves around the circle at its own individual frequency. When K is risen above a certain critical value (equal to $\frac{2}{\pi g(\Omega_0)}$, where Ω_0 can be determined by solving (13) for small values Kr) a group of oscillators with frequencies in the interval $[\Omega_0 - Kr, \Omega_0 + Kr]$ (r given by (12)) will synchronise. As K grows the size of the synchronised group grows. Dependent on the form of the distribution it is possible that another synchronised group arises, when K is increased. However, the previous calculations do not hold anymore for two different synchronised groups and the critical value for K for the appearance of the second group will probably be more difficult to derive. When K is risen further other groups can arise or groups can merge with each other into a synchronised group that will again grow with K. Although the previous calculations do not rigorously imply this process (the stability has not been investigated and only one synchronised group was considered) simulations (with a finite number of oscillators) indicate that this is what actually happens.

Example. We consider the distribution g defined by

$$g(\omega) = 2(1 - \omega), \quad \forall \omega \in [0, 1],$$

and zero outside $[0, 1]$. We can numerically solve equations (12) and (13) to obtain the relation between K and r as given in figure 1. (The critical value for K can be calculated to be $K_c \approx 0.4096$.) In figure 2 we keep the coupling strength fixed at $K = 0.42$ and we compare the long term average velocities of the oscillators for infinite

N with estimates obtained from simulations. For infinite N those values follow from the previous derivation (for this value of K we calculated $\Omega_0 \approx 0.2421$ and $Kr \approx 0.1999$). For the simulations we observed a population of 10000 oscillators (with individual frequencies randomly chosen out of the given distribution g) over a time interval of length $T = 1000$ with initial condition $\theta_i(0) = 0$, $\forall i$. We used the Euler method with a time-step of 0.1 and then plotted $\theta_i(T)/T$ versus ω_i for all i. Each different run is represented by a dashed line, the solid line represents the (numerically computed) infinite N case.

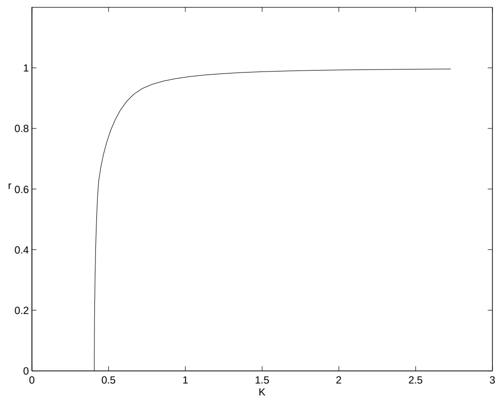

Figure 1: The order parameter r as a function of the coupling strength K.

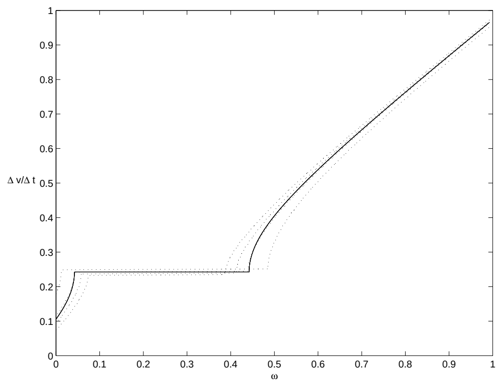

Figure 2: Long term average values of v as a function of ω. Solid line: infinite N case, numerically computed by (12) and (13); dashed lines: simulation results.

3. Oscillators with equal frequencies

One way to relate the result of the previous section to the finite N model is by considering the distribution g, with

$$g(\omega) = \frac{1}{N} \sum_{i=1}^{N} \delta(\omega - \omega_i). \qquad (14)$$

Since a δ-function has a maximum of infinity, application of the previous result would indicate that for small $K > 0$ there will be N synchronised groups, each corresponding to one of the ω_i and representing a fraction $\frac{1}{N}$ of the entire population. This does not mean that those oscillators move at a constant velocity. To know how the groups move one would still have to solve (1). This behaviour suggests that (for finite N), oscillators with the same frequencies will always synchronise. Of course, from the differential equations it follows for $\omega_i = \omega_j$ that if $\theta_i = \theta_j$ then $\dot{\theta}_i = \dot{\theta}_j$, implying that (on the circle) oscillator i and j can not pass each other, and thus their long term average frequencies will be the same. Simulations (with finite N) also show that if $\omega_i = \omega_j$ then $\theta_i - \theta_j \to 0$ (modulo 2π). Although this seems quite natural in such a system (where the interaction is attractive), even the local stability properties of the submanifold $\theta_i = \theta_j$ are already hard to investigate. However, for small $K > 0$ we have the following result (of which we omit the proof).

Proposition 1 *Assume that $\omega_i = \omega_j = \omega$ for some $i \neq j$, both in $\{1, \ldots, N\}$. Set $S = \{1, \ldots, N\} \setminus \{i, j\}$. If $\omega_k \neq \omega$, $\forall k \in S$, then there exists an $\epsilon > 0$, such that $\forall K \in (0, \epsilon)$ the submanifolds defined by $\theta_i = \theta_j + 2\pi m$, $m \in \mathbb{Z}$, are locally asymptotically stable under the flow of (1).*

This behaviour can be seen as a special case of partial entrainment, which is described in the next section.

4. Partial entrainment

While in the case of infinite N a solution may exist where a group of oscillators is moving at the same velocity (at all times), this is impossible for finite N (if all individual frequencies are different). However, there exist solutions where all oscillators within a group S_e have bounded phase differences, i.e.

$$\exists C > 0 : \left|\theta_i(t) - \theta_j(t)\right| < C, \quad \forall t \geq 0, \forall i, j \in S_e.$$

Definition. If this property holds for a non-empty proper subset of the population then we call the corresponding solution partially entrained with respect to this subset.

Note that according to this definition (which slightly differs from the one in [1]) there is always a trivial form of entrainment corresponding to the singletons $\{i\} \subset \{1, \ldots, N\}$. The following proposition gives a sufficient condition for the partial entrainment with respect to a subset S_e that contains more than half of the population.

Proposition 2 *Let S_e be a proper subset of $\{1, \ldots, N\}$ with M elements and such that $M > \frac{N}{2}$. Assume that*

$$\left|\omega_i - \omega_j\right| < K \sqrt{\frac{N}{M}} \left(\frac{4M - 2N}{3N}\right)^{\frac{3}{2}}, \quad \forall i, j \in S_e.$$

Then there exists a solution of (1) that is partially entrained with respect to S_e.

Proof

For some $a \in (0, \frac{\pi}{2}]$ let R_a denote the region

$$R_a = \{\theta \in \mathbb{R}^N : \left|\theta_i - \theta_j\right| \le a, \forall i, j \in S_e\}.$$

We will find a value for a for which R_a is a trapping region for (1). Assume that for some $t_0 \in \mathbb{R}$ the solution of (1) at time t_0 is located at the boundary of R_a: $\theta \in R_a$ and $\theta_i - \theta_j = a$ for some $i, j \in S_e$. From (1) it follows that

$$\dot{\theta}_i - \dot{\theta}_j = \omega_i - \omega_j - 2\frac{K}{N}\sin\left(\frac{\theta_i - \theta_j}{2}\right)\sum_{k=1}^{N}\cos\left(\theta_k - \frac{\theta_i + \theta_j}{2}\right).$$

In the summation we can bound the terms for which $k \in S_e$ by

$$\cos\left(\theta_k - \frac{\theta_i + \theta_j}{2}\right) \ge \cos a$$

since $\left|\theta_k - \frac{\theta_i+\theta_j}{2}\right| \le a$. (In fact $\left|\theta_k - \frac{\theta_i+\theta_j}{2}\right| \le \frac{a}{2}$, but this bound would result in more complicated calculations.) If $k \notin S_e$ we only know that $\cos\left(\theta_k - \frac{\theta_i+\theta_j}{2}\right) \ge -1$, and thus

$$\dot{\theta}_i - \dot{\theta}_j \le \omega_i - \omega_j - 2\frac{K}{N}\sin\frac{a}{2}\left(M\cos a - (N - M)\right).$$

For R_a to be a trapping region we need this to be negative. Minimising this expression by choosing a appropriately leads to $\sin\frac{a}{2} = \sqrt{\frac{2M-N}{6M}}$, resulting in

$$\dot{\theta}_i - \dot{\theta}_j \le \omega_i - \omega_j - K\sqrt{\frac{N}{M}}\left(\frac{4M - 2N}{3N}\right)^{\frac{3}{2}} < 0,$$

and thus for this value of a R_a is a trapping region. Since R_a is non-empty we can choose an initial condition in R_a and the resulting solution of (1) will exhibit partial entrainment with respect to S_e. □

On the other hand, if there are $i, j \in S_e$ with $\left|\omega_i - \omega_j\right| > 2K$, then partial entrainment with respect to S_e is impossible since $\theta_i - \theta_j$ will grow unboundedly. Of course, partial entrainment may still occur with respect to some other set. In some cases this allows to determine a maximal entrained subset, i.e. a subset for which partial entrainment can occur and which is not included in another entrained set. Consider for instance a system of 5 oscillators with $\omega_1 = -0.1$, $\omega_2 = 0$, $\omega_3 = 0.2$, $\omega_4 = 12$ and $\omega_5 = 100$. From the previous inequality and the result of the proposition it follows that for $K \in (4.78, 5.9)$ there exists a solution exhibiting partial entrainment for the set $\{1, 2, 3\}$, while all other entrainment sets must be subsets of $\{1, 2, 3\}$. For $K \in (42.78, 44)$ this is true for the set $\{1, 2, 3, 4\}$ and for $K > 183.9$ there exists a totally entrained solution.

Although these analytical results seem rather crude and forced us to take extreme values for the parameters, they are indicative for what happens in general and backed by simulation results we presume the following scenario to hold. If, for a generic system with N oscillators (and also for the previous example), the coupling strength K is increased from 0, N different ‘regimes’ can be distinguished. Each regime can be determined by a partition of $\{1, \ldots, N\}$ of which every element is maximal entrained subset (it may be a singleton). The different regimes correspond to successive intervals for K. First, for small values of K, there is no entrainment at all, all phase differences diverge for all possible solutions. Above some critical value for K almost all[1] solutions exhibit entrainment of two oscillators. When K is increased above the next critical value either two other oscillators form a new entrained subset or a third oscillator joins the previous one. This process continues and can generally be described as follows. For a fixed K almost all solutions show the same entrainment behaviour. When crossing a critical value for K (while increasing K) two entrained subsets (possibly singletons) merge into a new entrained subset, which is again the same for almost all initial conditions. After $N - 1$ transitions full entrainment occurs as described in [1].

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[1] ‘Almost all’ is to be interpreted in the following way. In the simulations *all* initial conditions give rise to the same behaviour. However, on analytical grounds one can see that there must be sets of initial conditions (probably with measure zero) for which the behaviour is different, i.e. for which the entrainment involves other subsets.