2005 International Symposium on Nonlinear Theory and its Applications (NOLTA2005)
Bruges, Belgium, October 18-21, 2005

Duality and the normalized left coprime factorization for a nonlinear system

Jacquelien M.A. Scherpen†

†Delft Center for Systems and Control, Delft University of Technology
Mekelweg 2, 2628 CD Delft, The Netherlands.
Email: j.m.a.scherpen@dcsc.tudelft.nl

Abstract—This paper considers the nonlinear left coprime factorization (NLCF) of a nonlinear system. In order to study the balanced realization of such NLCF first a dual system notion is introduced. The important energy functions for the original NLCF and their relation with the dual NLCF are studied and relations between these functions are established. These developments can be used for studying a relation between the singular value functions of the NLCF and the normalized right coprime factorization (NRCF) of a nonlinear system.

1. Introduction

In linear systems theory the Gramians of a system play an important role in many studies, and in especially when the study is dealing with balanced realizations. For unstable linear systems there exists balancing methods that are based on normalized coprime factorizations (e.g. [11], [8]). In those studies a relation between the Gramians of the right and left coprime factorizations and the solutions of the Control and Filter Algebraic Riccati Equation (CARE and FARE, respectively) are given.

A generalization for nonlinear systems is given in [15], where expressions for the normalized left and right coprime factorizations (NLCF and NRCF, respectively) are obtained. Other research, such as [1, 12] further developed coprime factorizations. In [15] the focus is mainly on balanced realizations for the NLCF and NRCF, and their relation with the HJB balanced representation.

In the case of NRCF [15] presents a similar relation as in the linear case for the observability and controllability function of the nonlinear NRCF, and the future and past energy function of the original nonlinear system (in the linear case, they correspond to the solutions of the CARE and FARE). However, a similar relation is not yet established in case of the NLCF. For that, we need to use a new notion of duality. The dual system as presented in this paper is inspired by the results in [6], where an adjoint state-space representation for the the nonlinear Hilbert adjoint, [14], is developed. For the dual system of the NLCF we are able to establish the relations that are similar to the ones for the NRCF.

Although the developments in this paper may also be important for nonlinear robust control (as in the linear case), our motivation for these developments stems from the nonlinear balanced realizations theory as a tool for model reduction. In [13, 5, 4] these tools are developed for asymptotically stable systems. It is well-known, [11, 8, 15] that for unstable systems the NRCF and NLCF can be considered.

In Section 2 we present some preliminaries about the NLCF of nonlinear systems. Then in Section 3 we establish relations for the dual system of the NLCF between the various energy functions that are important for balanced realizations. Finally in Section 4 we give some conclusions.

Notation: We denote $L_2(-\infty, 0)$ by L_2^- and $L_2(0,\infty)$ by L_2^+. Furthermore, by $\frac{\partial K}{\partial x}(x)$ we denote the row vector with partial derivative of a function $K(x)$.

2. Preliminaries

Consider a smooth nonlinear system of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \\ y &= h(x) \end{aligned} \tag{1}$$

where $u = (u_1, \ldots, u_m)^T \in \mathbb{R}^m$, $y = (y_1, \ldots, y_p)^T \in \mathbb{R}^p$, and $x = (x_1, \ldots, x_n)^T$ are local coordinates for a smooth state space manifold denoted by M. Furthermore, $f, g_1, \ldots, g_m$ are smooth vectorfields on M, where $g = (g_1, \ldots, g_m)$, and $h = (h_1, \ldots, h_p)^T$ is the smooth output map of the system. Throughout we assume that the system has an equilibrium. Without loss of generality we take this equilibrium in 0, i.e. $f(0) = 0$. We also take $h(0) = 0$.

We can relate several energy functions with system (1). This is done in the next definition.

Definition 2.1 *The controllability and observability function of a nonlinear system (1) are given by*

$$L_c(x_0) = \min_{\substack{u \in L_2^- \\ x(-\infty)=0, \\ x(0)=x_0}} \frac{1}{2}\int_{-\infty}^{0} \| u(t) \|^2 \, dt, \tag{2}$$

and

$$L_o(x_0) = \frac{1}{2}\int_{0}^{\infty} \| y(t) \|^2 \, dt, \quad x(0) = x_0, \quad u(t) \equiv 0, \quad 0 \le t < \infty, \tag{3}$$

respectively.
The past and future energy function of a nonlinear system

are defined as

$$K^-(x_0) = \min_{\substack{u \in L_2^- \\ x(-\infty)=0 \\ x(0)=x_0}} \frac{1}{2}\int_{-\infty}^{0} (\| y(t) \|^2 + \| u(t) \|^2) dt, \tag{4}$$

and

$$K^+(x_0) = \min_{\substack{u \in L_2^+ \\ x(\infty)=0 \\ x(0)=x_0}} \frac{1}{2}\int_{0}^{\infty} (\| y(t) \|^2 + \| u(t) \|^2) dt \tag{5}$$

respectively. □

The above energy functions are related to some Hamilton-Jacobi-Bellman type of equations, stemming from Optimal Control theory. First we give the equations for the observability and controllability function.

Theorem 2.2 *[13] Assume that $f(x)$ is asymptotically stable on a neighborhood W of 0. Then*

$$\frac{\partial \overline{L}_o}{\partial x}(x) f(x) + \frac{1}{2} h^T(x) h(x) = 0, \qquad \overline{L}_o(0) = 0. \tag{6}$$

has a smooth solution $\overline{L}_o$ for all $x \in W$. if and only if L_o exists. Then L_o is the unique smooth solution of (6) for all $x \in W$.
Furthermore, the Hamilton-Jacobi equation

$$\frac{\partial \overline{L}_c}{\partial x}(x) f(x) + \frac{1}{2}\frac{\partial \overline{L}_c}{\partial x}(x) g(x) g^T(x) \frac{\partial^T \overline{L}_c}{\partial x}(x) = 0, \tag{7}$$

$\overline{L}_c(0) = 0$ has a smooth solution $\overline{L}_c$ for all $x \in W$ such that

$$-(f(x) + g(x) g^T(x) \frac{\partial^T \overline{L}_c}{\partial x}(x)) \tag{8}$$

is asymptotically stable on W if and only if $L_c(x)$ exists. Then $L_c(x)$ is the unique smooth solution of (7), such that (8) is asymptotically stable, for all $x \in W$. □

Theorem 2.3 *e.g. [15] The Hamilton-Jacobi-Bellman equation*

$$\begin{aligned} \frac{\partial K^+}{\partial x}(x) f(x) \quad & -\frac{1}{2}\frac{\partial K^+}{\partial x}(x) g(x) g(x)^T \frac{\partial^T K^+}{\partial x}(x) \\ & +\frac{1}{2} h^T(x) h(x) = 0 \end{aligned} \tag{9}$$

with $K^+(0) = 0$, has a smooth non-negative solution on a neighborhood Y of 0, such that

$$f(x) - g(x) g(x)^T \frac{\partial^T K^+}{\partial x}(x) \tag{10}$$

is asymptotically stable, if and only if K^+ exists. Then K^+ is that solution.

Furthermore, the Hamilton-Jacobi-Bellman equation

$$\begin{aligned} \frac{\partial K^-}{\partial x}(x) f(x) \quad & +\frac{1}{2}\frac{\partial K^-}{\partial x}(x) g(x) g^T(x) \frac{\partial^T K^-}{\partial x}(x) \\ & -\frac{1}{2} h(x)^T h(x) = 0 \end{aligned} \tag{11}$$

with $K^-(0) = 0$, has a smooth non-negative solution on a neighborhood Y of 0, such that

$$-(f(x) + g(x) g(x)^T \frac{\partial^T K^-}{\partial x}(x)) \tag{12}$$

is asymptotically stable, if and only if K^- exists on Y. Then K^- is that solution. □

We assume the system (1) to be *zero-state observable*. Furthermore, we assume that (11) has a smooth non-negative solution K^- on a coordinate neighborhood Y of 0. It follows from (11) that $\frac{\partial K^-}{\partial x}(0) = 0$ and thus we can write (see [9])

$$\frac{\partial K^-}{\partial x}(x) = x^T M(x), \tag{13}$$

where $M(x)$ is an $n \times n$ matrix with all entries $m_{ij}(x)$, $i, j = 1, \ldots, n$, smooth functions of x and $M(0) = \frac{\partial^2 K^-}{\partial x^2}(0)$. We assume that

$$\frac{\partial^2 K^-}{\partial x^2}(0) > 0$$

and therefore there exists a neighborhood U of 0 for which $M(x)$ is nonsingular and thus is invertible on U. Furthermore, since $h(0) = 0$, we can write $h(x) = C(x)x$ where $C(x)$ is an $p \times n$ matrix with entries that are smooth functions of x and $C(0) = \frac{\partial h}{\partial x}(0)$. Now consider for $x \in U$

$$\begin{aligned} \dot{x} = & \; \left(f(x) - (M(x))^{-1} C(x)^T h(x) \right) + \\ & \; (g(x) \quad (M(x))^{-1} C(x)^T) \, \tilde{w} \\ z = & \; h(x) + (0 \quad -I) \, \tilde{w} \end{aligned} \tag{14}$$

This system is asymptotically stable on U under the assumption that K^- is proper on U. K^- then serves as a Lyapunov function for (14). The system (14) is a representation of the *normalized left coprime factorization* (NLCF) of (1), see [15, 12].

Remark 2.4 *It can be shown (see [15]) that linearizing the above system yields the corresponding linear NLCF. Since the linear NLCF is asymptotically stable, (14) is exponentially stable. Hence, there exists a neighborhood of 0 where all eigenvalues of $A(x) - (M(x))^{-1} C(x)^T C(x)$ are in the left half plane as well.* □

3. The NLCF and duality

For model reduction of nonlinear systems based on balanced realizations, see e.g. [13, 5, 3], the system has to be asymptotically stable. If this is not the case, we could consider to balance the normalized coprime factorization that is asymptotically stable. In [15] this is considered for the normalized right coprime factorization (NRCF), as well as for the nonlinear version of the linear LQG balancing (e.g., [7]), the so-called HJB balancing. For linear systems it does not matter if the NRCF or the NLCF is considered for balancing; the singular values are equal for the two factorizations. However, such relation is not established yet for nonlinear systems. Furthermore, the relation between the future and past energy functions and the controllability and observability functions of the NLCF is not established yet, whereas for NRCF this is already established in [15].

Now consider K^- and K^+ for the system (1) and the controllability and observability functions $\overline{L}_c$ and $\overline{L}_o$ for the NLCF given by (14). Then it is straightforwardly obtained that

$$K^-(x) = \overline{L}_c(x)$$

If we assume that system (1) is *linear*, and minimal, and thus also (14) is a linear system. Then we can write

$$\begin{aligned}\overline{L}_c(x) &= K^-(x) = \frac{1}{2}x^T Z x,\\ \overline{L}_o(x) &= \frac{1}{2}x^T X x, \quad K^+(x) = \frac{1}{2}x^T P x,\end{aligned}$$

where Z, X, and P are positive definite matrices, and for equation (13) we obtain $M(x) = Z$. Then Z^{-1} and X are the controllability and observability Gramian of the NLCF, respectively. Z^{-1} and P are the stabilizing solutions of the FARE (Filter Algebraic Riccati Equation) and CARE (Control Algebraic Riccati Equation), respectively, e.g., [11, 8]. Furthermore, it can be proven that those matrices are related via (e.g. [11])

$$Z^{-1} = X^{-1} - P^{-1} \tag{15}$$

Clearly, equation (15) is dealing with the inverses of the matrices that appear in the quadratic forms. This implies that (15) is not straightforwardly extended to the nonlinear case. In order to establish a relation like (15) for the nonlinear NLCF we first need to establish an appropriate notion of duality for nonlinear systems.

Now, we drop the assumption that the systems are linear, and we will consider a dual system that is inspired by the nonlinear Hilbert adjoint notion, [14] for which we have obtained state-space realizations in [6]. In [6] we mention duality "in the sense of Young", using the Legendre transformation of the controllability and observability functions. Here, we will use this notion related to the nonlinear Hilbert adjoint descriptions of [6].

Consider $f(x) = A(x)x$ and $h(x) = C(x)x$ as before, where $A(x)$, $C(x)$ are an $n \times n$ and $p \times n$ matrix with elements depending smoothly on x, with $A(0) = \frac{\partial f}{\partial x}(0)$ and $C(0) = \frac{\partial h}{\partial x}(0)$.

Now consider (1) in combination with the following dual system.

$$\begin{aligned}\dot{p} &= A(x)^T p + C(x)^T u_d\\ y_d &= g(x)^T p\end{aligned} \tag{16}$$

and consider the Legendre transform of $K^+(x)$ as follows:

$$\tilde{K}^+(p) = -K^+(x) + p^T x$$

then we can state the following lemma.

Lemma 3.1 *$\tilde{K}^+(p)$ fulfills the Hamilton-Jacobi-Bellman equation (11) for the past energy function of system (16).*

Proof: The result is straightforwardly obtained by considering equation (9) for system (1) and equation (11) for system (16) with $p = \frac{\partial^T K^+}{\partial x}(x)$. □

Remark 3.2 *Note that (16) is linear in p.* □

The dual system of the NLCF (14) is given by

$$\begin{aligned}\dot{p} &= \left(A(x)^T - C(x)^T C(x)\,(M(x))^{-1}\right) p\\ &\quad + C(x)^T \tilde{w}_d\\ z_d &= \begin{pmatrix} g(x)^T \\ C(x)\,(M(x))^{-1}\end{pmatrix} p + \begin{pmatrix} 0 \\ -I\end{pmatrix} \tilde{w}_d\end{aligned} \tag{17}$$

where x is a solution of (14).

If we consider the controllability function $\overline{L}_c(x)$ of (14), and its Legendre transform

$$\tilde{L}_c(p) = -\overline{L}_c(x) + p^T x,$$

then the corresponding dual coordinates are given by $p = \frac{\partial \overline{L}_c}{\partial x}(x) = \frac{\partial K^-}{\partial x}(x) = M(x)x$, and thus $x = M(x)^{-1}p$.

Lemma 3.3 *The Legendre transform of $\overline{L}_o(x)$, $\tilde{L}_o(p)$, fulfills the Hamilton-Jacobi-Bellman equation for the controllability function of system (17). Furthermore, $\tilde{L}_c(p)$ is the observability function of system (17).*

Proof: This follows immediately by considering the equations. □

Now we are able to establish the nonlinear counterpart of (15), i.e.,

Theorem 3.4 *With $p = M(x)x$ we have that $\tilde{L}_c(p) = \tilde{L}_o(p) - \tilde{K}^+(p)$.*

Proof: Consider the corresponding equations, i.e.,

$$\begin{aligned}\frac{\partial \tilde{L}_c}{\partial p}(p)\left(A(x)^T - C(x)^T C(x)\,(M(x))^{-1}\right) p&\\ + \tfrac{1}{2} p^T g(x) g(x)^T p&\\ + \tfrac{1}{2} p^T M(x)^{-1} C(x)^T C(x) M(x)^{-1} p &= 0\end{aligned}$$

$$\begin{aligned}\frac{\partial \tilde{L}_o}{\partial p}(p)\left(A(x)^T - C(x)^T C(x)\,(M(x))^{-1}\right) p&\\ + \tfrac{1}{2} \frac{\partial \tilde{L}_o}{\partial p}(p) C(x)^T C(x) \frac{\partial \tilde{L}_o}{\partial p}(p) &= 0\end{aligned}$$

$$
\begin{aligned}
&\frac{\partial \tilde{K}^+}{\partial p}(p)A(x)^T \\
&+\frac{1}{2}\frac{\partial \tilde{K}^+}{\partial p}(p)C(x)^T C(x)\frac{\partial^T \tilde{K}^+}{\partial p}(p) \\
&-\frac{1}{2}p^T g(x)g(x)^T p = 0
\end{aligned}
$$

Subtracting the equation for $\tilde{K}^+(p)$ from the equation for $\tilde{L}_o(p)$, where $p = M(x)x$, and thus $x = M(x)^{-1}p = \frac{\partial \tilde{L}_c}{\partial p}(p)$, the relation is established. □

Remark 3.5 *For a linear system Theorem 3.4 results in (15).* □

Due to linearity in p we can now easily write $\frac{\partial^T \tilde{L}_o}{\partial p}(p) = Y(x)p$ and $\frac{\partial^T \tilde{K}^+}{\partial p}(p) = W(x)p$, where $Y(x)$ and $W(x)$ are positive definite matrices on $x \in U$.

Corollary 3.6 *The linearity in p yields*

$$x = Y(x)\frac{\partial^T L_c}{\partial x}(x) - W(x)\frac{\partial^T L_c}{\partial x}(x)$$

Proof: Since

$$x = \frac{\partial^T \tilde{L}_c}{\partial p}(p) = \frac{\partial^T \tilde{L}_o}{\partial p}(p) - \frac{\partial \tilde{K}^+}{\partial p}(p)$$

and $p = \frac{\partial^T L_c}{\partial x}(x)$, we obtain the result. □

Remark 3.7 *For linear systems we have that $Y(x) = X^{-1}$ and $W(x) = P^{-1}$, and thus Corollary 3.6 yields*

$$x = X^{-1}Zx - P^{-1}Zx,$$

which results in (15). □

4. Concluding remarks

In this paper we have studied the controllability, observability and past and future energy functions of the normalized left coprime factorization of a nonlinear system. Furthermore, the notion of duality in the sense of Young, related to the notion of nonlinear Hilbert adjoints, [6], has been used in order to establish the relations between the respective functions. The considered functions are important for balancing the coprime factorizations, which on its turn is a useful tool for model reduction of unstable nonlinear systems.

References

[1] J. A. Ball and A. J. van der Schaft. *J*-inner-outer factorization, *J*-spectral factorization, and robust control for nonlinear systems. *IEEE Trans. Autom. Contr.*, AC-41(3):379–392, 1996.

[2] P. E. Crouch and A. J. van der Schaft. *Variational and Hamiltonian Control Systems*, volume 101 of *Lecture Notes on Control and Information Science*. Springer-Verlag, Berlin, 1987.

[3] K. Fujimoto and J. M. A. Scherpen. Input-normal/output-diagonal forms and model reduction for nonlinear systems based on the differential eigenstructure of Hankel operators. *Proc. 40th IEEE Conf. Dec. Contr.*, Orlando, Florida, 3252–3257, 2001.

[4] K. Fujimoto and J. M. A. Scherpen. Nonlinear balanced realization based on the differential eigenstructure of Hankel operators. *Proc. Conf. Dec. Contr. CDC03*, Hawai, USA, 6072–6077, 2003 .

[5] K. Fujimoto and J. M. A. Scherpen. Nonlinear input-normal realizations based on the differential eigenstructure of Hankel operators. *IEEE Trans. Aut. Contr.*, AC-50(1):2–18, 2005.

[6] K. Fujimoto, J. M. A. Scherpen, and W. S. Gray. Hamiltonian realizations of nonlinear adjoint operators. *Automatica*, 38(10):1769–1775, 2002.

[7] E. A. Jonckheere and L. M. Silverman. A new set of invariants for linear systems - Applications to reduced order compensator design. *IEEE Trans. Aut. Contr.*, AC-28:954–964, 1983.

[8] D.G. Meyer. Fractional balanced reduction - model reduction via a fractional representation. *IEEE Trans. Aut. Contr.*, AC-26:1341-1345, 1990.

[9] J. Milnor. *Morse Theory*. Annals of Math. Stud. 51, Princeton University Press, 1963.

[10] B. C. Moore. Principal component analysis in linear systems: controllability, observability and model reduction. *IEEE Trans. Autom. Contr.*, AC-26:17–32, 1981.

[11] R. Ober and D. McFarlane. Balanced canonical forms for minimal systems: A normalized coprime factor approach. *Lin. Alg. Appl.*, 23–64, 1989.

[12] A. D. P. Paice and A. J. van der Schaft. The class of stabilizing nonlinear plant controller pairs. *IEEE Trans. Autom. Contr.* AC-41:634–645, 1996.

[13] J. M. A. Scherpen. Balancing for nonlinear systems. *Systems & Control Letters*, 21:143–153, 1993.

[14] J. M. A. Scherpen and W. S. Gray. Nonlinear Hilbert Adjoints: Properties and Applications to Hankel Singular Value Analysis. *Nonlinear Analysis, Theory, Methods and Applications*, 51(5):883–901, 2002.

[15] J. M. A. Scherpen and A. J. van der Schaft. The normalized left and right coprime factorization and balancing for nonlinear systems. *Int. J. Contr.*, 60:1193–1222, 1994.