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Abstract—We demonstrate that time-delayed feedback control can be improved by adaptively tuning the feedback parameters, using the speed-gradient method of control theory. For a delay-coupled network of Stuart-Landau oscillators (normal form of supercritical Hopf bifurcation) we show that by adaptively tuning the coupling phase, as well as the coupling strength and the delay time, one can easily control the stability of different synchronous periodic states, namely in-phase, cluster, or splay states. Our results are robust even for slightly nonidentical elements of the network.

1. Introduction

Time delays arise naturally in many complex systems and networks, for instance in neural networks or coupled lasers, as delayed coupling or delayed feedback due to finite signal transmission and processing times. Such time delays can either induce instabilities, multistability, and complex bifurcations, or suppress instabilities and stabilize unstable states. Thus, they can be used to control the dynamics [1, 2]. Here we propose to use adaptive control schemes based on optimizations of cost or goal functions [3, 4] to find appropriate control parameters. In particular, we apply adaptive control to synchronization in networks [5]. The existence and stability of various synchronous states, i.e., in-phase, cluster, or splay states, in networks of delay-coupled Stuart-Landau oscillators was studied by Choe *et al.* [6]. This Stuart-Landau system arises naturally as a generic expansion in complex variables in the center manifold near a Hopf bifurcation and is therefore often used as a paradigm for oscillators. The complex coupling constant that occurs in networks of Stuart-Landau oscillators consists of an amplitude and a phase. Such phase-dependent couplings have also been shown to be important in overcoming the odd-number limitation of time-delay feedback control [7, 8, 9] and in anticipating chaos synchronization [10]. Furthermore, it was shown in Refs. [6] that the value of the coupling phase is a crucial control parameter in these systems, and by adjusting this phase one can deliberately switch between different synchronous oscillatory states of the network.

2. Network model

Consider a network of N delay-coupled oscillators

$$\dot{z}_j(t) = f[z_j(t)] + Ke^{i\beta} \sum_{n=1}^{N} a_{jn}[z_n(t-\tau) - z_j(t)] \quad (1)$$

with $z_j = r_j e^{i\varphi_j} \in \mathbb{C}$, $j = 1, \dots, N$. The coupling matrix $A = \{a_{ij}\}_{i,j=1}^{N}$ determines the topology of the network. The local dynamics of each element is given by the normal form of a supercritical Hopf bifurcation, also known as Stuart-Landau oscillator,

$$f(z_j) = [\lambda + i\omega - (1 + i\gamma)|z_j|^2]z_j \quad (2)$$

with real constants $\lambda, \omega \neq 0$, and γ. In Eq. (1), τ is the delay time. K and β denote the amplitude and phase of the complex coupling constant, respectively.

Synchronous in-phase (or zero-lag), cluster, and splay states are possible solutions of Eqs. (1) and (2). They exhibit a common amplitude $r_j \equiv r_{0,m}$ and phases given by $\varphi_j = \Omega_m t + j\Delta\varphi_m$ with a phase shift $\Delta\varphi_m = 2\pi m/N$ and collective frequency Ω_m. The integer m determines the specific state: in-phase oscillations correspond to $m = 0$, while splay and cluster states correspond to $m = 1, \dots, N-1$. The cluster number d, which determines how many clusters of oscillators exist, is given by the least common multiple of m and N divided by m, and $d = N$ (e.g., $m = 1$), corresponds to a splay state.

The stability of synchronized oscillations in networks can be determined numerically, for instance, by the *master stability function* [11]. This formalism allows for a separation of the local dynamics of the individual nodes from the network topology. Within the master stability approach general properties of synchronization in delay-coupled networks have been discussed for zero-lag synchronization and large delay [12], and for cluster or group synchronization [13]. In the case of the Stuart-Landau oscillators it is possible to obtain the Floquet exponents of different cluster states analytically with this technique [6]. By these means it has been demonstrated that the unidirectional ring configuration of Stuart-Landau oscillators exhibits in-phase synchrony, splay states, and clustering depending on the choice of the control parameter β. For $\beta = 0$, there exists multistability of the possible synchronous states in a large parameter range. However, for certain values of the coupling phase $\beta = \Omega_m\tau - 2\pi m/N$ according to a particular state m, this synchronous state can be shown analytically to be monostable for any values of the coupling strength K and the time delay τ. In the following, an adaptive algorithm is used to find optimal values of the control parameters K, β, and τ by automatic adaptive adjustment.

3. Speed-gradient method

We use the speed-gradient (SG) method [3] as an adaptive control scheme for the general nonlinear dynamical system

$$\dot{x} = F(x, u, t) \tag{3}$$

with state vector $x \in \mathbb{C}^n$, input (control) variables $u \in \mathbb{C}^m$, and nonlinear function F. Define a control goal

$$\lim_{t\to\infty} Q(x(t), t) = 0, \tag{4}$$

where $Q(x, t) \geq 0$ is a smooth scalar goal function.

In order to design a control algorithm, the scalar function $\dot{Q} = \omega(x, u, t)$ is calculated, that is, the speed (rate) at which $Q(x(t), t)$ is changing along trajectories of Eq. (3). Then a differential equation is set up for the self-adaptive adjustment of the input variables u

$$\frac{du}{dt} = -\Gamma\nabla_u\omega(x, u, t), \tag{5}$$

where $\Gamma = \Gamma^{\mathrm{T}} > 0$ is a positive definite gain matrix.

The idea of this algorithm is the following. The term $-\nabla_u\omega(x, u, t)$ points to the direction in which the value of $\dot{Q}$ decreases with the highest speed. Therefore, if one forces the control signal to "follow" this direction, the value of $\dot{Q}$ will decrease and finally be negative. When $\dot{Q} < 0$, then Q will decrease and, eventually, tend to zero.

We shall now apply the speed-gradient method to networks of Stuart-Landau oscillators. First, since the coupling phase β is the crucial parameter that determines stability of the possible in-phase, cluster, and splay states, we use this control parameter as the input variable u. Setting $u = \beta$, $x = (z_1, \ldots, z_N)$ and $\Gamma = \Gamma_\beta$, Eq. (1) takes the form of Eq.(3) with state vector $x \in \mathbb{C}^N$ and input variable $\beta \in \mathbb{R}$, and nonlinear function $F(x, \beta, t) = [f(z_1), \ldots, f(z_N)] + Ke^{i\beta}[Ax(t-\tau) - x(t)]$.

The SG control equation (5) for the input variable β then becomes

$$\frac{d\beta}{dt} = -\Gamma_\beta\frac{\partial}{\partial\beta}\omega(x, \beta, t) = -\Gamma_\beta\left(\frac{\partial F}{\partial\beta}\right)^{\mathrm{T}}\nabla_x Q(x, t), \tag{6}$$

where $\Gamma_\beta > 0$ is now a scalar.

4. Zero-lag synchronization

To apply the SG method for the selection of in-phase (zero-lag) synchronization we need to find an appropriate goal function Q. It should satisfy the following conditions: the goal function must be zero for an in-phase synchronous state and larger than zero for other states. Hence, a simple goal function can be introduced by considering a function based on the Kuramoto order parameter

$$R_1 = \frac{1}{N}\left|\sum_{j=1}^{N} e^{i\varphi_j}\right|. \tag{7}$$

It is obvious that $R_1 = 1$ if and only if the state is in-phase synchronized. For other cases we have $R_1 < 1$. Using this observation we can introduce the following goal function

$$Q_0 = 1 - \frac{1}{N^2}\sum_{j=1}^{N} e^{i\varphi_j}\sum_{k=1}^{N} e^{-i\varphi_k}. \tag{8}$$

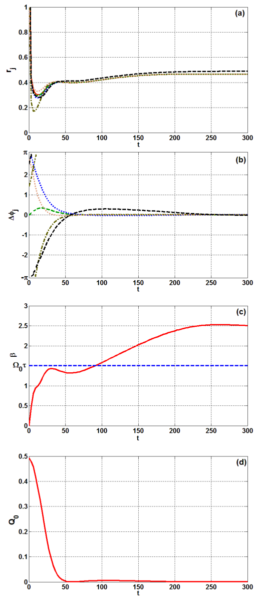

Figure 1: Adaptive control of in-phase oscillations. (a) $r_j = |z_j|$; (b) $\Delta\phi_j = \varphi_j - \varphi_{j+1}$; (c) temporal evolution of β; (d) goal function.

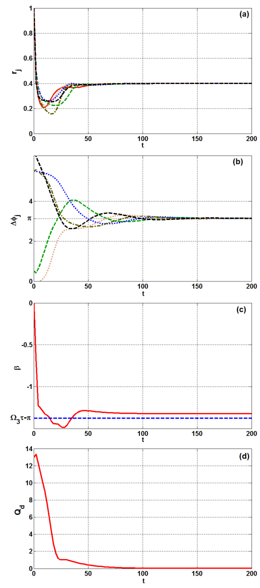

Figure 2: Same as in Fig. 1 for adaptive control of 2-cluster state ($m = 3$).

From $\dot{\beta} = -\Gamma_\beta\frac{\partial}{\partial\beta}\dot{Q}_0$ we derive an adaptive law:

$$\dot{\beta} = \Gamma_\beta\frac{2K}{N^2}\sum_{k,j=1}^{N}\sin(\varphi_k - \varphi_j)\sum_{n=1}^{N} a_{jn}\left(\frac{r_{n,\tau}}{r_j}\cos(\beta + \varphi_{n,\tau} - \varphi_j) - \cos\beta\right). \tag{9}$$

Fig. 1 shows the results of a numerical simulation for an Erdős-Rényi random network with $N = 6$ nodes and row sum normalized to unity. Unless otherwise stated, we use $\Gamma_\beta = 1$. According to the numerical simulations decreasing Γ_β will yield a decrease of the speed of convergence. On the other hand, if Γ_β is too big, undesirable oscillations appear. The model parameters are chosen as in Ref. [6] ($\lambda = 0.1, \omega = 1, \gamma = 0$, $K = 0.08$, $\tau = 0.52\pi$, $N = 6$).

Initial conditions for r_j and φ_j are chosen randomly from $[0,4]$ and $[0,2\pi]$, respectively, $\beta(0) = 0$. The amplitudes and phases approach appropriate values that lead to in-phase synchronization. Note that the obtained value of β does not converge to the one for which the analytical approach [6] has established stability of the in-phase oscillation (blue dashed line), but to another limit value. This can be explained as follows: There exists a whole interval of acceptable values of β around the value of the coupling phase for which an analytical treatment is possible, such that for any value from this interval an in-phase state is stable. Our SG algorithm finds one of them, depending upon initial conditions.

5. Cluster synchronization

In this section we will consider unidirectionally coupled rings with $N = 6$ nodes. Let $1 \le m \le N-1$. Then $d = \mathrm{LCM}(m,N)/m$, where LCM denotes the least common multiple, is the number of different clusters of a synchronized solution. In order to extend the goal function Eq. (8) such that we can stabilize d-cluster states, we define a generalized order parameter

$$R_d = \frac{1}{N}\left|\sum_{k=1}^{N} e^{di\varphi_k}\right| \tag{10}$$

with $d \in \mathbb{N}$. However, if we derive a goal function from this order parameter in an analogous way as in Eq. (8), this function will not have a unique minimum at the d-cluster state because $R_d = 1$ holds also for the in-phase state and for other p-cluster states where p are divisors of d. Therefore, we adopt the following goal function:

$$Q_d = 1 - f_d(\varphi) + \frac{N^2}{2}\sum_{p|d,1\le p<d} f_p(\varphi), \tag{11}$$

where $p|d$ means that p is a factor of d. This goal function contains f_d as the primary contribution for the d-cluster state, but also a sum of penalty terms that counteract reaching other cluster states in which f_d is also unity. Whenever one of those unwanted cluster states is approached, the penalty term will lead to a gradient away from it. The prefactor $N^2/2$ is chosen for convenience to secure faster convergence of the algorithm. From $\dot\beta = -\Gamma_\beta \frac{\partial}{\partial\beta}\dot Q_d$ one can derive the adaptation law

$$\begin{aligned}\dot\beta \;=\; & -\Gamma_\beta K \sum_{j,k=1}^{N}\left\{\sum_{p|d,1\le p<d} p\sin[p(\varphi_k-\varphi_j)] - \frac{2d}{N^2}\sin[d(\varphi_k-\varphi_j)]\right\} \\ & \times \sum_{n=1}^{N} a_{jn}\left[\frac{r_{n,\tau}}{r_j}\cos(\beta+\varphi_{n,\tau}-\varphi_j) - \cos(\beta)\right].\end{aligned} \tag{12}$$

Figs. 2 and 3 depict the results of numerical simulations for two clusters ($d = 2$, $m = 3$, $N = 6$) and three clusters ($d = 3$, $m = 2$ or 4), respectively. We note that the obtained value of β comes close to the one (blue dashed line) for which stability was shown analytically in Ref. [6].

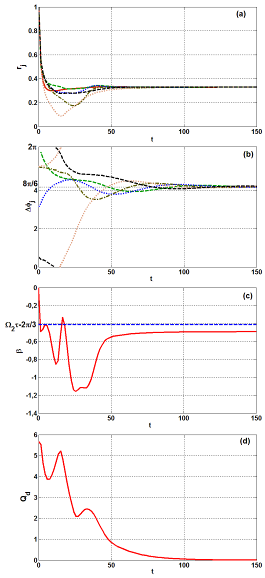

Figure 3: Same as in Fig. 1 for adaptive control of 3-cluster state ($m = 2, 4$).

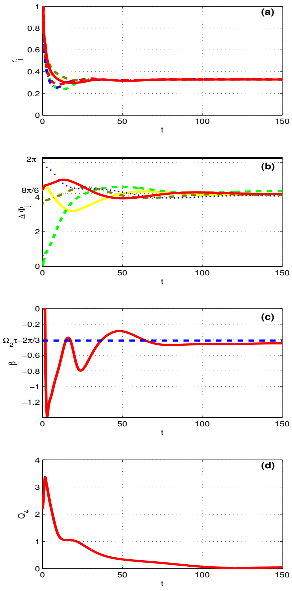

Figure 4: Same as in Fig. 1 for the 3-cluster state in a network with nonidentical oscillators.

All the results presented so far were for identical oscillators in the network. It has been shown that control of cluster and splay states using an appropriate value of the phase β works even for slightly nonidentical frequencies ω of the oscillators [6]. Fig. 4 shows the adaptive control of a 3-cluster state similar to Fig. 3, but with nonidentical parameters λ_j and ω_j of the individual oscillators. We choose them from a Gaussian distribution with mean value $\lambda = 0.1$ and $\omega = 1$, respectively, and standard deviation 1% for both. The oscillators do not synchronize completely due to their amplitude and frequency mismatch, which can be seen in Fig. 4(a,b).

6. Controlling several parameters simultaneously

The general form of the SG method as given in Eq. (5) is also suitable for controlling more than one parameter. In this section, this is demonstrated by controlling β, K, and τ simultaneously. The vector u in Eq. (5) is then given by $u = (\beta, K, \tau)$. We choose Γ as a diagonal matrix with the diagonal elements $\Gamma_{11} \equiv \Gamma_\beta$, $\Gamma_{22} \equiv \Gamma_K$ and $\Gamma_{33} \equiv \Gamma_\tau$. Using the goal function Q_d of Eq. (11) we obtain for β the adaptive algorithm given by Eq. (12). For $\dot K = -\Gamma_K \frac{\partial}{\partial K}\dot Q_d$ we obtain:

$$\begin{aligned}\dot K \;=\; & -\Gamma_K \sum_{j,k=1}^{N}\left\{\sum_{p|d,1\le p<d} p\sin[p(\varphi_k-\varphi_j)] - \frac{2d}{N^2}\sin[d(\varphi_k-\varphi_j)]\right\} \\ & \times \sum_{n=1}^{N} a_{jn}\left[\frac{r_{n,\tau}}{r_j}\sin(\beta+\varphi_{n,\tau}-\varphi_j) - \sin(\beta)\right].\end{aligned} \tag{13}$$

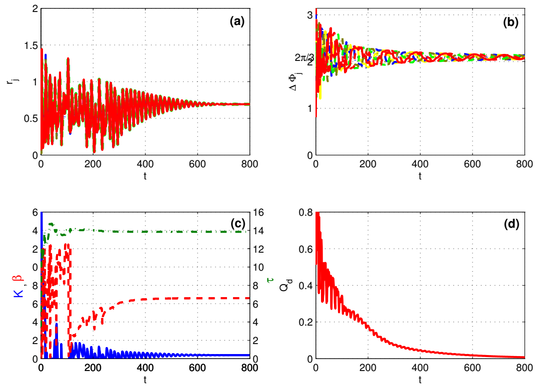

Figure 5: Same as in Fig. 1 for adaptive control of 3-cluster state in a network of 6 nodes by simultaneously tuning K, β and τ according to Eqs. (12), (14), and (13), respectively. $\Gamma_\beta = \Gamma_K = 10$, $\Gamma_\tau = 0.05$.

and for $\dot{\tau} = -\Gamma_\tau \frac{\partial}{\partial \tau} \dot{Q}_d$

$$\dot{\tau} = -\Gamma_\tau \sum_{j,k=1}^{N} \left\{ \sum_{p|d,1\leq p<d} p \sin[p(\varphi_k - \varphi_j)] - \frac{2d}{N^2} \sin[d(\varphi_k - \varphi_j)] \right\}$$
$$\times \sum_{n=1}^{N} a_{jn} \left[-\frac{\dot{r}_{n,\tau}}{r_j} \sin(\beta + \varphi_{n,\tau} - \varphi_j) - \dot{\varphi}_{n,\tau} \frac{r_{n,\tau}}{r_j} \cos(\beta + \varphi_{n,\tau} - \varphi_j) \right]. \quad (14)$$

Figure 5 shows the successful control of a 3-cluster state in a network consisting of 6 nodes where appropriate values of β, K, and τ are found adaptively.

7. Conclusion

We have shown that by combining time-delayed coupling with the speed gradient method of control theory one can adaptively control synchronization in oscillator networks, Choosing an appropriate goal function, a desired state of generalized synchrony can be selected by the self-adaptive automatic adjustment of a control parameter. This goal function, which is based on a generalization of the Kuramoto order parameter, vanishes for the desired state, e.g., in-phase or various cluster states. By numerical simulations we have shown that those different states can be stabilized, and the control parameter (coupling phase) converges to an appropriate value. We have established the robustness of the control scheme by investigating slightly nonidentical oscillators. We have also applied our method simultaneously to the coupling phase, coupling amplitude, and the time delay. In this way control of cluster synchronization is possible without any a priori knowledge of the coupling parameters. Given the paradigmatic nature of the Stuart-Landau oscillator as a generic model, we expect broad applicability, for instance to synchronization of networks in medicine, chemistry or mechanical engineering. The mean-field nature of our goal function makes our approach accessible even for very large networks independently of the particular topology.

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