IEICE Proceeding Series

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Vol. 1 pp. 574-577
Publication Date: 2014/03/17
Online ISSN: 2188-5079

2012 International Symposium on Nonlinear Theory and its Applications
NOLTA2012, Palma, Majorca, Spain, October 22-26, 2012

Co-evolving Network Dynamics between Reaction-Diffusive Resources on Nodes and Weighted Connections

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Abstract—We investigated the co-evolving dynamics of a network and the state on it. We considered a reaction-diffusion system on a weighted network, in which a dissipative resource on the nodes, such as molecules, individuals, money, or data packets, move diffusively to other nodes through weighted links. Simultaneously, the weighted connections dynamically change in a resource-dependent manner. We demonstrate that this interplay between dynamics both on and of a network, yields self-organized network of the interaction of the dynamical system, involving an emergence of power-law distributions in both the quantities of the resource and the strengths of the links. Our results offer a framework for understanding the functional structures of real-world networks pertinent to resource distribution.

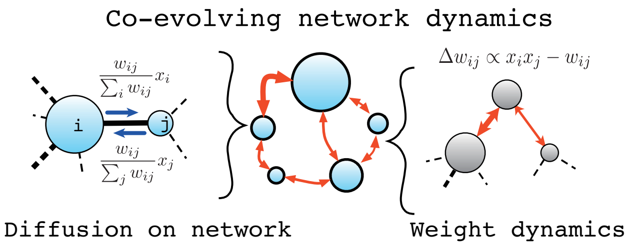

Figure 1: Schematic illustration of co-evolving network dynamics, in which the link weights of the network and dynamical states of the nodes influence each other. In our model, the quantity of resource x_i of the ith node is transported diffusively to the connected nodes through the weighted links w_{ij}, while the weights of the links evolve in a resource-dependent manner (i.e., the law of mass action).

1. Introduction

The term *network* is commonly used in a wide range of research fields, including physics, mathematics, biology, computer science, engineering, and sociology. The word usually indicates complex relationships of the interactions observed in these research fields. In general, it is difficult to interpret structure of such complex interactions and the mechanisms that organize them.

In this study, we propose a model of network organization that integrates both the dynamics of a network and the state on it. In most previous network science studies, the structure of the interactions is simply described by an adjacency matrix a_{ij}, in which a_{ij} =1 or 0 depending on whether the link between nodes i and j exists or not, respectively. This abstract representation has provided a general framework to analyze network feature observed in the real-world networks, and to explain the network organization process of them. For example, the Barabási-Albert (BA) algorithm [1], which generates a model of scale-free networks, is based only on information from an adjacency matrix, such as the degree distribution. Most other models of network organization also have been discussed within the framework of an adjacency matrix, even though several extensions have been considered [2, 3, 4, 5]. These simple models that represent a network abstractly are beneficial for understanding the essential mechanisms of complex real-world networks. However, there is another important aspect of real-world networks, i.e., network links originate from relationships of the interactions in the system, and the organization of these interactions depends directly on the dynamical state of the system. For example, in a neural network, the modulation of the synaptic connection between neurons depends directly on the activities of the neurons, and not on the topology of the neural network [6, 7], even though the topology of the network represented in the adjacency matrix can affect the organization process indirectly, since it affects the dynamics of the system. In other biological, social, and technological networks, the organization of a network is usually dependent on local information of the activities of the system, while the topology of the network actually affects these system activities [8]. Therefore, the interplay between the dynamical states of the systems on a network and the topological evolution of the network of the interactions is a key for obtaining a more complete understanding of the network organization process for these interactions. Therefore, we formulate a co-evolving network dynamics, integrating both types of dynamics.

2. Model

In this study, we propose a framework for mathematical modeling of co-evolving weighted networks, by em-

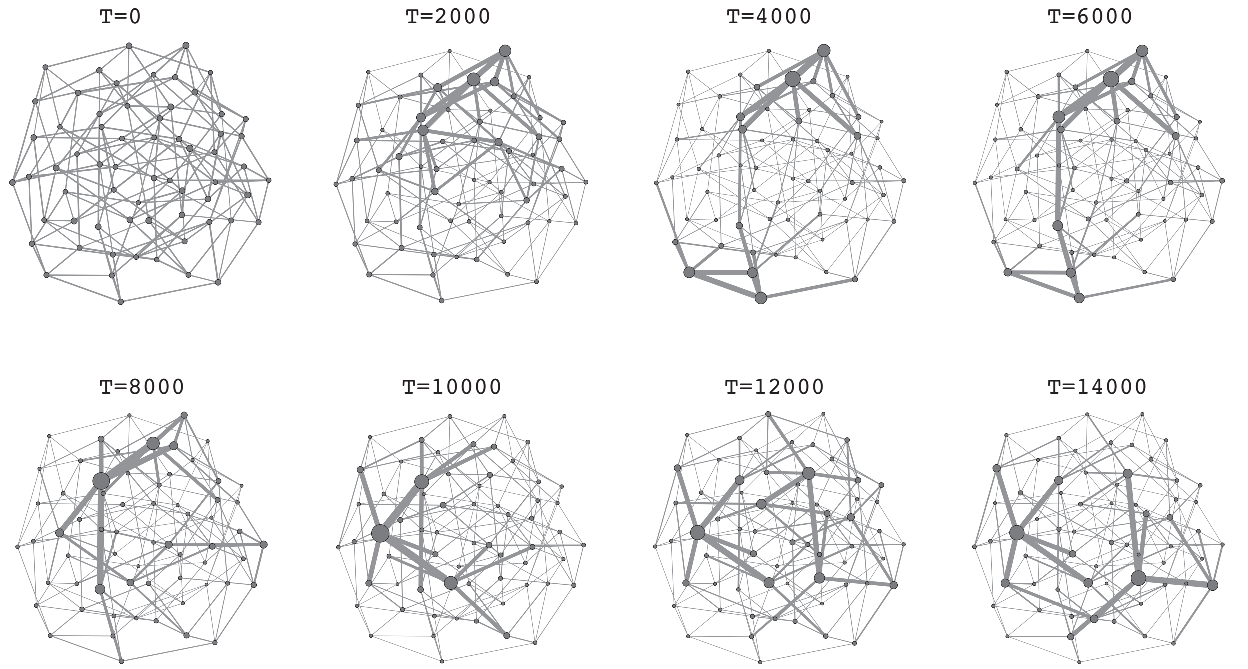

Figure 2: Time development of the quantity of the resource at nodes and the weights of the connections, organized through the co-evolving dynamics defined by equations (1) and (2). In the graph, the size of the circle represents the quantity of the resource at the node, and the width of the link represents the weight of the link. The initial values of the resource and the weight are generated with mean $\mu = 1$ and standard deviation $\sigma = 0.1$. The underlying topology is given by a regular random graph of size $N = 64$ and degree $k = 5$). The other parameter values are $\epsilon = 0.01$, $\kappa = 0.05$, and $D = 0.35$.

ploying reaction-diffusion dynamics. Diffusion is a very simple but fundamental process in many physical and social phenomena occurring in real-world networks, such as traffic flows and transports over the network, information dissemination on communication networks, and epidemic spreading. Therefore, among the possible dynamical processes on a network, we here consider a diffusion-based dynamical process on a network as illustrated in Fig.1. We consider the quantity delivered by the diffusion process as a resource for the network. It may correspond in realistic situations, for example, molecules, cells, people, or money. The quantity of the resource represents the state x_i of the i-th node in the network. We formulate the reaction diffusion dynamics of the resource as follows. First, let us consider a weighted network of N nodes. The link structure of the network is defined by the adjacency matrix a_{ij}. We assign a time-dependent weight $w_{ij}(t)$ to each existing link. This weight represents the strength of the interaction. In addition, we assume that these weights are symmetric; that is, $w_{ij}(t) = w_{ji}(t)$. The evolution of $x_i(t)$ is given by the following equation:

$$\Delta x_i(t) = F(x_i(t)) + \text{diffusion process via links},$$

where $\Delta x_i(t) \equiv x_i(t+1) - x_i(t)$. $F(x)$ represents the reaction process of the resource for which we employ a simple dissipative with equilibrium $x = 1$, described by $F(x) = -\kappa(x - 1)$. The diffusion process can be understood as the diffusion of many random walkers, in which the walkers at node i move to the neighboring node j in accordance with the time-dependent weighted probability $Dw_{ji}(t)/s_i(t)$. Here, $s_i(t)$ is the strength of the node i defined by $s_i(t) \equiv \sum_{j \in \mathcal{N}_i} w_{ji}(t)$, where $\mathcal{N}_i$ is the set of nodes connected with node i. Thus, the master equation for the resource is given by

$$\Delta x_i(t) = F(x_i(t)) + D \sum_{j \in \mathcal{N}_i} \left(\frac{w_{ij}}{s_j} x_j - \frac{w_{ji}}{s_i} x_i \right), \qquad (1)$$

where the second and third terms are the inward and outward current of the resources at i-th node, respectively.

Simultaneously, the network is organized according to the resource distribution between the nodes. Even though little is known about the elementary process of network organization in real-world networks, in this study, we introduce a simple dynamics of the link weights in accordance with the law of mass action, as follows. We assume that the evolution of the weight $w_{ij}(t)$ depends on the two resources at the endpoint nodes of the link, $x_i(t)$ and $x_j(t)$. At the first step, we assume that the quantities of the resource at the two nodes have linear dependency, so the weight $w_{ij}(t)$ merely relaxes to $x_i(t)x_j(t)$. Therefore, the dynamics of the weights is described by

$$w_{ij}(t+1) - w_{ij}(t) = \epsilon \left[x_i(t)x_j(t) - w_{ij}(t) \right], \qquad (2)$$

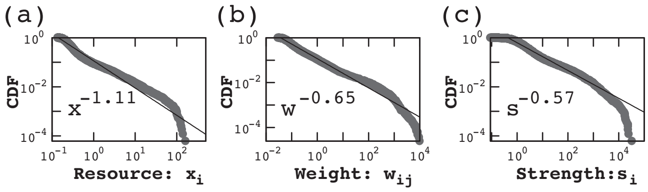

Figure 3: Typical network features emerging through the co-evolving dynamics. Cumulative distributions of the resources x_i, weights w_{ij}, and strengths s_i converge to power-law forms. The initial topology was chosen to be the Erdös-Rényi random graph with $N = 16384$ and $\langle k \rangle = 10$. The initial resources and weights were generated by a normal distribution with mean $\mu = 1$ and standard deviation $\sigma = 0.1$. Other parameters are $\kappa = 0.05$, $D = 0.34$, and $\epsilon = 0.01$.

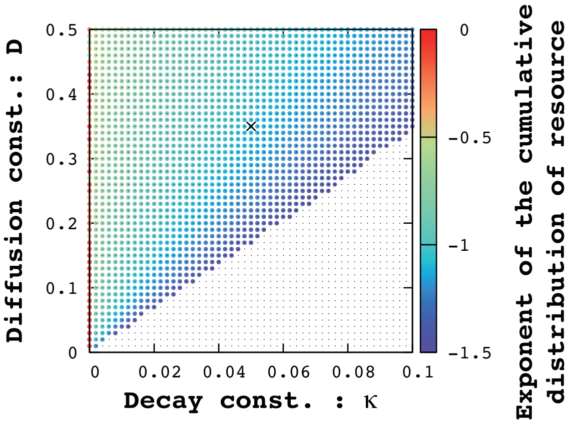

Figure 4: Dependence of the exponent of the cumulative distribution of resources on the decay constant, κ, and the diffusion constant, D. The other parameter values are the same as in Fig. 3. The "x" indicates the parameter values used in Fig. 3.

where the parameter ϵ^{-1} represents the relaxation time scale of the weight dynamics.

3. Results

Figure 2 shows the time development of a typical behavior of the co-evolving dynamics defined by equations (1) and (2). In the graphs of the networks, the size of the circle and width of the link represent the quantity of the resource at the node and weight of the connection, respectively. Their initial values are almost homogeneous, and are generated by a normal distribution with mean $\mu = 1$ and standard deviation $\sigma = 0.1$. From the time step $t = 0$ to $t = 4000$, the quantities of the resource are gathered into the few nodes at the top of the graph, and simultaneously, the weights of the connections between these resource-rich nodes become potentiated. In other words, a heterogeneous network of the resource and the weight is organized. Next, until $t = 6000$, other nodes (at the bottom of the graph) begin to develop their quantities of the resource, and then the path between the resource-rich nodes at the top and bottom becomes potentiated. The potentiation of this path causes the resource distribution among the nodes to change drastically. At $t = 8000$, the resource is concentrated in the nodes located on the way to the path. In contrast, the nodes that were hubs at previous times lose their resource and the potentiated weights of the links from other nodes. Then, similar processes are repeated. The resource distribution among nodes and the weights of the links between them continue to change indefinitely through the co-evolving dynamics.

For a large network ($N = 16384$), we investigated the cumulative distributions of the resource and the weight. We have found that, under feasible conditions, these distributions asymptotically converge to power-law forms as shown in Fig. 3. The cumulative distribution of the resource takes a power-law form with exponent $\gamma \sim -1$. This distribution is consistent with Zipf's law [9]. This type of power-law distribution has been reported in many physical and social phenomena, including word frequencies in natural languages, populations of cities, statistics on Web access, and the sizes of companies [10, 11, 12, 13, 14].

Similarly, the weights w_{ij} also exhibit a power-law distribution in the stationary state with a different exponent. In this stationary state, we confirmed that these distributions converge to fixed forms, even though on the microscopic level, the the resource quantities and weights change indefinitely as stated above. Moreover, strengths s_i ($\equiv \sum_j a_{ij} w_{ij}$), which corresponds to the generalized metric of degree k ($\equiv \sum_j a_{ij}$) in weighted networks, also exhibit a power-law distribution. Therefore, through the dynamics, a type of scale-free network can be organized.

The exponent of the resource power-law distribution generally depends on the parameter values, such as the decay constant, κ, and the diffusion constant, D. Figure 4 plots the exponent of the cumulative resource distribution in (κ,D) space. These results were obtained by fitting the numerically generated distributions to the form x^{γ} with least-squares fit. In the graph, the color of a point on the grid indicates the value of its exponent, corresponding to the color bar at right. As seen there, the exponent decreases to approximately -1.5 with increasing κ and decreasing D. In a regime at high κ and low D, for which there are no colored points on the grid, the organized distributions are of a non-power law type. Thus, we see that the resource disparity among the nodes is an increasing function of D and a decreasing function of κ.

4. Summary

In this study, employing reaction-diffusion dynamics, we have introduced a framework for mathematical modeling of co-evolving weighted networks and have investigated the system in a simple dissipative diffusion process of a single resource with resource-dependent weight evolution (in accordance with the law of mass action). We have demonstrated that both the resource and weight distributions exhibit power-law forms in the asymptotic state as a result of the interplay between those dynamics both on and of the network. Furthermore, we have found a dynamical phase in an organized scale-free network.

These findings provide novel insight of the functioning of the network structure, because an organized network structure is directly related to resource diffusion on a network through co-evolving dynamics. Therefore, we believe that further study of co-evolving dynamics will help us elucidate functional network structures pertinent to resource distribution over networks.

Acknowledgments

This study was supported by Grants-in-Aid from the Ministry of Education, Science, Sports, and Culture of Japan: Grant numbers 24120708, 24740266, 21120002, and 23115511.

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