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Bifurcations in coupled Hamiltonian networks

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Abstract—We present recent results from Chan et al (*Dynamical Systems: an international journal*, **32**, 2016) on networks of coupled Hamiltonian systems and discuss the Hamiltonian-Hopf theorem in this context. We then present a version of the Lyapunov centre theorem for these types of networks.

1. Introduction

In recent years, several formalisms have been put forward to study coupled networks of differential equations. The groupoid formalism developed by Golubitsky, Stewart and several co-authors [13, 4, 5, 12, 15, 14] has shown, amongst other results, that these networks support non-generic types of bifurcation phenomena. Rink and Sanders [9, 10] have shown that for homogeneous regular networks, those can be given a semi-group structure which enables the use of standard techniques from semigroup representation theory to study local bifurcations. The network may also have additional structure. For instance, Manoel and Roberts [6] studied whether a network can be regarded as a gradient system and provide an application of their theory to the Kuramoto model and the Antiferromagnetic XY model. In a similar direction, Chan *et al.* [3] look at the problem of coupling Hamiltonian systems in a network and examine whether the network structure preserves the Hamiltonian structure. They also present results on linear theory near an equilibrium solution and state some local bifurcation results. In this short paper, we review some of the results of [3]. In particular, the necessary and sufficient condition for a homogeneous regular network with linear coupling to be Hamiltonian, some linear theory results and the Hamiltonian-Hopf theorem. We conclude with a new result which expresses the Lyapunov Centre Theorem in the context of coupled Hamiltonian networks.

2. Hamiltonian coupled cell networks

Coupled cell systems are collections of N ordinary differential equations called *cells* with phase space variable $x_i \in \mathbb{R}^{k_i}$, for $i \in \{1, \ldots, N\}$. Suppose that cell i receives input from cells $j_{i_1}, \ldots, j_{i_m}$, then the dynamics of the i^{th} cell can be written as

$$\frac{dx_i}{dt} = f_i(x_i, x_{j_{i_1}}, \ldots, x_{j_{i_m}}).$$

Coupled cell systems can be represented graphically using directed graphs (or digraphs) which consist of a *vertex set* V and an *arc set* E where each arc is an ordered pair of distinct vertices. See Figure 1 for an example with $V = \{v_1, v_2, v_3\}$ and $E = \{e_1, e_2, e_3, e_4\}$ where $e_1 = \{2, 1\}$, $e_2 = \{1, 2\}$, $e_3 = \{3, 1\}$ and $e_4 = \{2, 3\}$. The connectivity of a graph G is

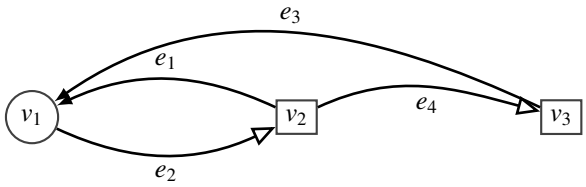

Figure 1: Example of a digraph representing a coupled cell system.

encoded in its *adjacency matrix* $A(G)$ which is an integer matrix with rows and columns indexed by the vertices of G, such that $A(G)[i, j]$ is equal to the number of arcs from cell i to j. We use straight edges for bidirectional coupling.

A coupled cell system is *homogeneous* if all cells have phase space of the same dimension. Also, a coupled cell system is *regular* if all coupling functions are identical. In particular, a regular homogeneous coupled cell system has the same number of inputs to each cell, also called the *valency* of the network. In this paper, we focus on homogeneous regular coupled cell systems.

A differential equation $\dot{x} = g(x)$ with $x \in \mathbb{R}^{2n}$ is said to be *Hamiltonian* if there exists a smooth function $H : \mathbb{R}^{2n} \to \mathbb{R}$ such that $g(x) = J\nabla H(x)$ where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

with I_n the $n \times n$ identity matrix. We look at homogeneous regular coupled cell systems of Hamiltonian differential equations where each cell has dimension $2n$ and written in the form

$$\frac{dx_i}{dt} = g(x_i, x_{i_1}, \ldots, x_{i_m}) \tag{1}$$

for $i = 1, \ldots, N$ where $g(x_i, 0, \ldots, 0) = J\nabla H(x_i)$ for some smooth function $H : \mathbb{R}^{2n} \to \mathbb{R}$ implies each cell is a Hamiltonian system. Note that the equal number of inputs to each cell constraints the form of g to have a fixed number of en-

tries. We assume linear coupling of the coupled cell system (1) by imposing the condition

$$D_{x_{i_k}} g(0,0,\ldots,0) = \mathcal{R} \tag{2}$$

for $k = 1,\ldots,m$ where $\mathcal{R}$ is a $2n \times 2n$ nonzero constant matrix called the *coupling matrix*. One can represent the coupling structure of the coupled cell system by the matrix $R = A \otimes \mathcal{R}$ where $\otimes$ is the Kronecker product of matrices. Note that a $2n \times 2n$ matrix M is *Hamiltonian* if $M^T J + JM = 0$. We can now recall the first result.

Theorem 2.1 (Chan *et al.* [3]) *Consider the homogeneous regular coupled cell system (1) with linear coupling (2) and for which each cell is Hamiltonian. Then, the coupled cell system is Hamiltonian if and only if the coupling matrix $\mathcal{R}$ is Hamiltonian and the adjacency matrix A is symmetric.*

The proof of this result is done by splitting the system into linear and nonlinear terms. Since the coupling is linear, the nonlinear terms are the same for all cells. The Hamiltonian structure of $\mathcal{R}$ and the symmetry of the adjacency matrix are necessary and sufficient conditions for the linearized coupled cell system to be Hamiltonian. One can then construct the Hamiltonian function for the full system easily because of the uniformity of the nonlinear terms across cells.

Example 2.2 *Consider the coupled cell system given in Figure 2 with valency four. Note that this network has*

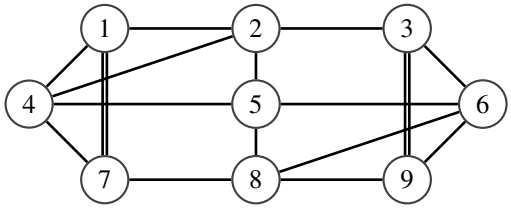

Figure 2: Network Γ_1.

Abelian symmetry group $\mathbf{D}_2$ *generated by the reflections* (2 6)(1 9)(4 8) *and* (2 4)(3 7)(6 8)*. The adjacency matrix is*

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \tag{3}$$

and has simple eigenvalues $4, \pm 2\sqrt{2}$ *and double eigenvalues* $\pm 1, 2$*. Because the symmetry group is Abelian, the double eigenvalues are not a consequence of the symmetry. For instance, the eigenvectors of* -2 *are* $V_1 = (1,0,-1,0,0,0,-1,0,1)$ *and* $V_2 = (0,1,-1,1,-2,1,-1,1,0)$.

3. Linear theory

We consider system (1) to satisfy Theorem 2.1 so that A is symmetric and $\mathcal{R}$ is Hamiltonian. We assume without loss of generality that system (1) has an equilibrium solution at the origin; that is $g(0,0,\ldots,0) = 0$. We let $Q = D_{x_i} g(0,0,\ldots,0)$ be the linearization at the origin of the dynamics of cell i. It is also a $2n \times 2n$ matrix. Then, the linearization of (1) at the origin can be written as

$$\mathcal{M} = I_N \otimes Q + A \otimes \mathcal{R}. \tag{4}$$

We now review some results from Golubitsky and Lauterbach [15]. Let $\mu_1,\ldots,\mu_s$ be distinct eigenvalues of the adjacency matrix A and consider the $2n \times 2n$ matrix $\mathcal{M}_{\mu_i} = Q + \mu_i \mathcal{R}$. Then, a first result states that the $2nN$ eigenvalues of $\mathcal{M}$ are the union of eigenvalues of the matrices $\mathcal{M}_{\mu_i}$ for $i = 1,\ldots,s$. In particular, if $\mu \in \mathbb{R}$ is an eigenvalue of A with associated eigenspace $E_A(\mu) \subset \mathbb{R}^N$, then $E_A(\mu) \otimes \mathbb{C}^{2n}$ is invariant under $\mathcal{M}$.

In order to discuss the Hamiltonian-Hopf bifurcation in the next section, we need to determine the eigenspace structure if eigenvalues on the imaginary axis collide as a parameter is being varied. Collisions of eigenvalues in Hamiltonian matrices can be sorted out using the Krein signature (see [7]) and for the case of purely imaginary eigenvalues two cases occur generically: the $1:1$ resonance and the $1:-1$ resonance. Only the $1:-1$ resonance gives rise to bifurcations, see [1].

We now identify the situation where a pair of purely imaginary eigenvalues in $1:-1$ arises as a parameter is varied. Fom a matrix $\mathcal{M}$ defined by (4), we say that $(Q,\mathcal{R})$ is a *Hamiltonian codimension-one pair* if all the eigenvalues of $\mathcal{M}_{\mu_1},\ldots,\mathcal{M}_{\mu_s}$ are distinct or exactly one such matrix $\mathcal{M}_{\mu_j}$ for some $j = 1,\ldots,s$ has either one semisimple zero eigenvalue of multiplicity two or one purely imaginary eigenvalue in $1:-1$ resonance. We have the following result.

Proposition 3.1 (Chan *et al.*[3]) *Suppose that $(Q,\mathcal{R})$ is a Hamiltonian codimension-one pair with $\sigma = i\omega$ in $1:-1$ resonance be an eigenvalue of $\mathcal{M}_\mu$ where μ is an eigenvalue of the adjacency matrix A. Let $\mathcal{G}_\mathcal{M}(\sigma)$ be the generalized eigenspace of σ in $\mathcal{M}$, then*

$$\mathcal{G}_\mathcal{M}(\sigma) \simeq E_A(\mu) \oplus E_A(\mu) \oplus E_A(\mu) \oplus E_A(\mu)$$

where $E_A(\mu)$ is the eigenspace of μ and their exists a basis of $E_A(\mu) \otimes \mathbb{C}^{2n}$ such that

$$\mathcal{M}\Big|_{\mathcal{G}_\mathcal{M}(\sigma)} = \begin{pmatrix} 0 & -\omega & 1 & 0 \\ \omega & 0 & 0 & 1 \\ 0 & 0 & 0 & -\omega \\ 0 & 0 & \omega & 0 \end{pmatrix} \otimes I_p.$$

The proof of this theorem shows that the basis of $\mathcal{G}_\mathcal{M}(\sigma)$ can be written as

$$\bigoplus_{j=1}^{p} \operatorname{span}\{V_j \otimes \operatorname{Im}(U_1), V_j \otimes \operatorname{Re}(U_1), V_j \otimes \operatorname{Im}(U_2), V_j \otimes \operatorname{Re}(U_2)\} \tag{5}$$

where $V_1, \ldots, V_p$ is a basis for $E_A(\mu)$.

We remark that the cell dimension needed to have purely imaginary eigenvalues in $1:-1$ resonance in M_μ must be at least 4. Note that the versal unfolding of the $1:-1$ resonance is

$$\begin{pmatrix} 0 & -\omega & 1 & 0 \\ \omega & 0 & 0 & 1 \\ \lambda & 0 & 0 & -\omega \\ 0 & \lambda & \omega & 0 \end{pmatrix} \otimes I_p.$$

4. The Hamiltonian-Hopf Theorem

The bifurcation of periodic solutions occurring at a $1:-1$ resonance in generic systems of Hamiltonian equations is described by the Hamiltonian Hopf Theorem [16]. In the context of a Hamiltonian network of coupled Hamiltonian cells, Proposition 3.1 provides the linear structure at the equilibrium solution from which we can establish the bifurcation result. However, we must identify the subspaces where the $1:-1$ resonance can be applied and for which synchronization between some cells is achieved.

For a graph G with vertex set $V(G) = N$, the phase space P has dimension $2nN$ and we write the coordinates as $x = (x_1, \ldots, x_N)$ where x_j is an element of cell j. Here, we define concepts introduced in Stewart *et al* [13] about the subspace structure of coupled cell systems. We denote by $\bowtie$ an equivalence relation on G and define a *polydiagonal subspace* associated with $\bowtie$ as

$$\Delta_{\bowtie} = \{x \in P \mid x_i = x_j \text{ whenever } i \bowtie j,\ \forall i, j \in \{1, \ldots, N\}\}.$$

The subspace $\Delta_{\bowtie}$ is flow-invariant for all admissible vector fields for the network structure given by G if $\bowtie$ is a "balanced" equivalence relation. The exact definition of balanced equivalence relation requires a lengthy discussion and we refer the reader to [13] for all the details. For homogeneous and regular networks, it can be described as follows. For an equivalence relation $\bowtie$, all cells in the same equivalence class are given the same colour, different from the other equivalence classes. An equivalence relation $\bowtie$ for a homogeneous regular coupled cell system is said to be *balanced* if for every cells c, d in the vertex set V of G such that $c \bowtie d$, then cells c and d receive the same number of inputs from cells of the same colour. The balanced equivalence relation concept is therefore crucial to determine subspaces for which the dynamics on the coupled cell system is flow-invariant. We say that $\Delta_{\bowtie}$ is a *synchrony subspace* if $\bowtie$ is balanced. In our context, we not only need the flow-invariance of the synchrony subspace, but we must make sure that the Hamiltonian structure is preserved too. This is proved in [3]. A synchrony subspace Δ is also a symplectic subspace and if H is the Hamiltonian function for the vector field on P, then $H|_\Delta$ is the Hamiltonian function for the vector field restricted to Δ.

A Hamiltonian in $\mathbb{R}^4$ with $1:-1$ resonance as studied by van der Meer [16] has a normal form given by the Hamiltonian function $H(x, y) = S + N + \mu P + aP^2$ where $S = x_1y_2 - x_2y_1$, $N = \frac{1}{2}(x_1^2 + x_2^2)$, $P = \frac{1}{2}(y_1^2 + y_2^2)$ The sign of a determines the appearance of the periodic solutions according to two different scenarios as described in [16, 1]. Write system (1) as $\dot{x} = F(x, \lambda)$. Suppose that $dF(0, 0)$ has $\pm i\omega$ eigenvalues in $1:-1$ resonance and E is its generalized eigenspace, the following result was established.

Theorem 4.1 (Chan *et al* [3]) *Consider a 1-parameter family of Hamiltonian networks of coupled Hamiltonian cells with an equilibrium solution at the origin. Suppose the linearization at the origin has a $1:-1$ resonance with generalized eigenspace E. Let Δ be a synchrony subspace such that $\dim(\Delta \cap E) = 4$. Let a_Δ be the coefficient of the normal form of H_0 on $\Delta \cap E$. Then, provided $a_\Delta \neq 0$, the same two scenarios occur as for the ordinary Hamiltonian Hopf bifurcation theorem.*

We illustrate Theorem 4.1 using the network of Example 2.2.

Example 4.2 *Balanced equivalence relations are obtained by coloring the cells in different colors. Note that each cell of the same color, receives the same number of inputs from a given category of color of cells. See Figure 3 for the equivalence relation leading to the synchrony subspace Δ_2 below. For instance, here, each black cell receives two inputs from white cells and two inputs from dark grey cells. We obtain synchrony subspaces $\Delta_1 = \{(a, b, c, b, d, b, c, b, a) \mid a, b, c, d \in \mathbb{R}^k\}$, $\Delta_2 = \{(a, b, a, c, d, c, a, b, a) \mid a, b, c, d \in \mathbb{R}^k\}$. As described in [3], if $\sigma = i\omega$ occurs for the matrix M_μ where μ is one of the double eigenvalues of A, then let U_1 be the eigenvector of σ and U_2 a generalized eigenvector and let V_1, V_2 be a basis of a double eigenvalue -2 of A. Using the formula (5) one can see that $\mathcal{G}_M(\mu) \subset \Delta_1$ because $V_1, V_2 \in \Delta_1$ as can be easily verified. Thus, the periodic orbit bifurcating has the synchronization pattern of Δ_1.*

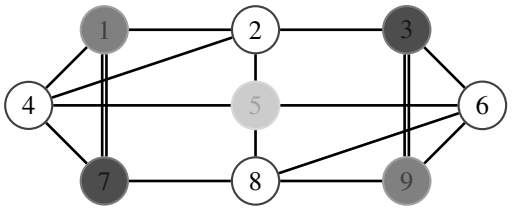

Figure 3: The Δ_1 synchrony subspace in Γ_1.

5. The Lyapunov Centre Theorem

We conclude this short article by considering an extension of the Lyapunov Centre Theorem to the case of Hamiltonian networks. This result is not found in [3]. The Lyapunov centre theorem describes the existence of a family of periodic orbits in the neighborhood of an equilibrium

solution with a pair of nonresonant purely imaginary eigenvalues [11] such that the frequency of the family of orbits converges to the frequency given by the purely imaginary eigenvalue. This result has been extended to the equivariant [8] and reversible-equivariant setting [2]. Let (1) be written as $\dot{x} = F(x)$ and assume $F(0) = 0$. Recall from [11] that eigenvalues λ of Hamiltonian matrices come in quadruplets $\lambda, \overline{\lambda}, -\lambda, -\overline{\lambda}$. We have the following result.

Theorem 5.1 *Suppose that $dF(0)$ has a pair of purely imaginary eigenvalues $\lambda_1 = i\omega$ in $\mathcal{M}_\mu$ such that all other eigenvalues λ (except $\overline{\lambda}_1$) are nonresonant; that is, $\lambda \neq k\lambda_1$ for any integer k. Let E be the eigenspace of $\lambda_1, \overline{\lambda}_1$ and Δ be a synchrony subspace such that* $\dim(\Delta \cap E) = 2$. *Then, there exists a* 1*-parameter family of periodic orbits γ_ϵ for $\epsilon \in [0, \epsilon_0]$ in $\Delta \cap E$ forming a smooth two-dimensional manifold of periodic orbits containing the equilibrium $x = 0$ and such that the period of γ_ϵ converges to $2\pi/\omega$ as $\epsilon \to 0$.*

Proof: The assumptions of the theorem guarantee that the Hamiltonian subsystem in the synchrony subspace Δ satisfies the assumptions of the standard Lyapunov Centre Theorem [11]. The conclusion follows immediately. ∎

Example 5.2 *Suppose $\lambda_1 = i\omega \in \mathcal{M}_{-1}$ with complex eigenvector U_1. The -1 eigenvalue of A in Example 2.2 has eigenvalue $W_1 = (0, 1, 0, -1, 0, -1, 0, 1, 0) \in \Delta_2$. Then, span$\{W_1 \otimes \mathrm{Im}(U_1), W_1 \otimes \mathrm{Re}(U_1)\} \subset \Delta_2$ and the periodic orbits have the synchronization pattern given in Figure 4.*

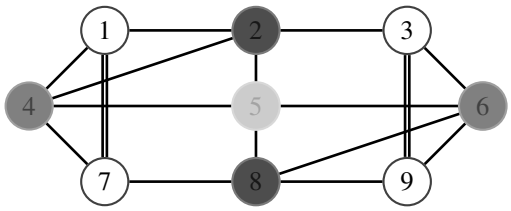

Figure 4: The Δ_2 synchrony subspace in Γ_1.

Acknowledgments

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