

Bifurcations in coupled Hamiltonian networks

Pietro-Luciano Buono[†], Bernard Chan[‡] and Antonio Palacios[‡]

[†]Faculty of Science, University of Ontario Institute of Technology, Oshawa, ONT, L1H 7K4, Canada

[‡]Department of Mathematics, San Diego State University, San Diego, CA, USA

Email: luciano.buono@uoit.ca

Abstract—We present recent results from Chan et al (*Dynamical Systems: an international journal*, **32**, 2016) on networks of coupled Hamiltonian systems and discuss the Hamiltonian-Hopf theorem in this context. We then present a version of the Lyapunov centre theorem for these types of networks.

1. Introduction

In recent years, several formalisms have been put forward to study coupled networks of differential equations. The groupoid formalism developed by Golubitsky, Stewart and several co-authors [13, 4, 5, 12, 15, 14] has shown, amongst other results, that these networks support non-generic types of bifurcation phenomena. Rink and Sanders [9, 10] have shown that for homogeneous regular networks, those can be given a semi-group structure which enables the use of standard techniques from semi-group representation theory to study local bifurcations. The network may also have additional structure. For instance, Manoel and Roberts [6] studied whether a network can be regarded as a gradient system and provide an application of their theory to the Kuramoto model and the Antiferromagnetic XY model. In a similar direction, Chan *et al.* [3] look at the problem of coupling Hamiltonian systems in a network and examine whether the network structure preserves the Hamiltonian structure. They also present results on linear theory near an equilibrium solution and state some local bifurcation results. In this short paper, we review some of the results of [3]. In particular, the necessary and sufficient condition for a homogeneous regular network with linear coupling to be Hamiltonian, some linear theory results and the Hamiltonian-Hopf theorem. We conclude with a new result which expresses the Lyapunov Centre Theorem in the context of coupled Hamiltonian networks.

2. Hamiltonian coupled cell networks

Coupled cell systems are collections of N ordinary differential equations called *cells* with phase space variable $x_i \in \mathbb{R}^{k_i}$, for $i \in \{1, \dots, N\}$. Suppose that cell i receives input from cells j_1, \dots, j_m , then the dynamics of the i^{th} cell can be written as

$$\frac{dx_i}{dt} = f_i(x_i, x_{j_1}, \dots, x_{j_m}).$$

Coupled cell systems can be represented graphically using directed graphs (or digraphs) which consist of a *vertex set* V and an *arc set* E where each arc is an ordered pair of distinct vertices. See Figure 1 for an example with $V = \{v_1, v_2, v_3\}$ and $E = \{e_1, e_2, e_3, e_4\}$ where $e_1 = \{2, 1\}$, $e_2 = \{1, 2\}$, $e_3 = \{3, 1\}$ and $e_4 = \{2, 3\}$. The connectivity of a graph G is

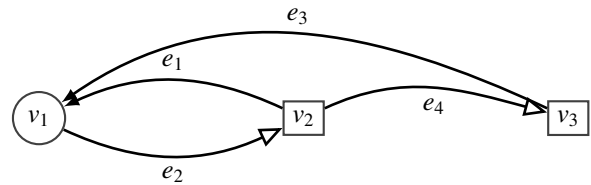


Figure 1: Example of a digraph representing a coupled cell system.

encoded in its *adjacency matrix* $A(G)$ which is an integer matrix with rows and columns indexed by the vertices of G , such that $A(G)[i, j]$ is equal to the number of arcs from cell i to j . We use straight edges for bidirectional coupling.

A coupled cell system is *homogeneous* if all cells have phase space of the same dimension. Also, a coupled cell system is *regular* if all coupling functions are identical. In particular, a regular homogeneous coupled cell system has the same number of inputs to each cell, also called the *valency* of the network. In this paper, we focus on homogeneous regular coupled cell systems.

A differential equation $\dot{x} = g(x)$ with $x \in \mathbb{R}^{2n}$ is said to be *Hamiltonian* if there exists a smooth function $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ such that $g(x) = J\nabla H(x)$ where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

with I_n the $n \times n$ identity matrix. We look at homogeneous regular coupled cell systems of Hamiltonian differential equations where each cell has dimension $2n$ and written in the form

$$\frac{dx_i}{dt} = g(x_i, x_{j_1}, \dots, x_{j_m}) \quad (1)$$

for $i = 1, \dots, N$ where $g(x_i, 0, \dots, 0) = J\nabla H(x_i)$ for some smooth function $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ implies each cell is a Hamiltonian system. Note that the equal number of inputs to each cell constraints the form of g to have a fixed number of en-

tries. We assume linear coupling of the coupled cell system (1) by imposing the condition

$$D_{x_k} g(0, 0, \dots, 0) = \mathcal{R} \quad (2)$$

for $k = 1, \dots, m$ where \mathcal{R} is a $2n \times 2n$ nonzero constant matrix called the *coupling matrix*. One can represent the coupling structure of the coupled cell system by the matrix $R = A \otimes \mathcal{R}$ where \otimes is the Kronecker product of matrices. Note that a $2n \times 2n$ matrix M is *Hamiltonian* if $M^T J + JM = 0$. We can now recall the first result.

Theorem 2.1 (Chan et al. [3]) Consider the homogeneous regular coupled cell system (1) with linear coupling (2) and for which each cell is Hamiltonian. Then, the coupled cell system is Hamiltonian if and only if the coupling matrix \mathcal{R} is Hamiltonian and the adjacency matrix A is symmetric.

The proof of this result is done by splitting the system into linear and nonlinear terms. Since the coupling is linear, the nonlinear terms are the same for all cells. The Hamiltonian structure of \mathcal{R} and the symmetry of the adjacency matrix are necessary and sufficient conditions for the linearized coupled cell system to be Hamiltonian. One can then construct the Hamiltonian function for the full system easily because of the uniformity of the nonlinear terms across cells.

Example 2.2 Consider the coupled cell system given in Figure 2 with valency four. Note that this network has

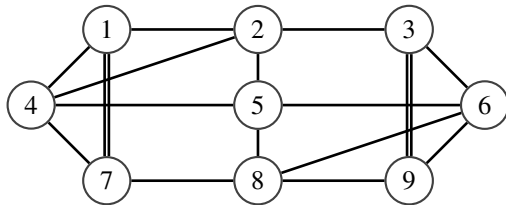


Figure 2: Network Γ_1 .

Abelian symmetry group \mathbf{D}_2 generated by the reflections $(2\ 6)(1\ 9)(4\ 8)$ and $(2\ 4)(3\ 7)(6\ 8)$. The adjacency matrix is

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \quad (3)$$

and has simple eigenvalues $4, \pm 2\sqrt{2}$ and double eigenvalues $\pm 1, 2$. Because the symmetry group is Abelian, the double eigenvalues are not a consequence of the symmetry. For instance, the eigenvectors of -2 are $V_1 = (1, 0, -1, 0, 0, 0, -1, 0, 1)$ and $V_2 = (0, 1, -1, 1, -2, 1, -1, 1, 0)$.

3. Linear theory

We consider system (1) to satisfy Theorem 2.1 so that A is symmetric and \mathcal{R} is Hamiltonian. We assume without loss of generality that system (1) has an equilibrium solution at the origin; that is $g(0, 0, \dots, 0) = 0$. We let $Q = D_{x_i} g(0, 0, \dots, 0)$ be the linearization at the origin of the dynamics of cell i . It is also a $2n \times 2n$ matrix. Then, the linearization of (1) at the origin can be written as

$$\mathcal{M} = I_N \otimes Q + A \otimes \mathcal{R}. \quad (4)$$

We now review some results from Golubitsky and Lauterbach [15]. Let μ_1, \dots, μ_s be distinct eigenvalues of the adjacency matrix A and consider the $2n \times 2n$ matrix $\mathcal{M}_{\mu_i} = Q + \mu_i \mathcal{R}$. Then, a first result states that the $2nN$ eigenvalues of \mathcal{M} are the union of eigenvalues of the matrices \mathcal{M}_{μ_i} for $i = 1, \dots, s$. In particular, if $\mu \in \mathbb{R}$ is an eigenvalue of A with associated eigenspace $E_A(\mu) \subset \mathbb{R}^N$, then $E_A(\mu) \otimes \mathbb{C}^{2n}$ is invariant under \mathcal{M} .

In order to discuss the Hamiltonian-Hopf bifurcation in the next section, we need to determine the eigenspace structure if eigenvalues on the imaginary axis collide as a parameter is being varied. Collisions of eigenvalues in Hamiltonian matrices can be sorted out using the Krein signature (see [7]) and for the case of purely imaginary eigenvalues two cases occur generically: the $1 : 1$ resonance and the $1 : -1$ resonance. Only the $1 : -1$ resonance gives rise to bifurcations, see [1].

We now identify the situation where a pair of purely imaginary eigenvalues in $1 : -1$ arises as a parameter is varied. From a matrix \mathcal{M} defined by (4), we say that (Q, \mathcal{R}) is a *Hamiltonian codimension-one pair* if all the eigenvalues of $\mathcal{M}_{\mu_1}, \dots, \mathcal{M}_{\mu_s}$ are distinct or exactly one such matrix \mathcal{M}_{μ_j} for some $j = 1, \dots, s$ has either one semisimple zero eigenvalue of multiplicity two or one purely imaginary eigenvalue in $1 : -1$ resonance. We have the following result.

Proposition 3.1 (Chan et al. [3]) Suppose that (Q, \mathcal{R}) is a Hamiltonian codimension-one pair with $\sigma = i\omega$ in $1 : -1$ resonance be an eigenvalue of \mathcal{M}_{μ} where μ is an eigenvalue of the adjacency matrix A . Let $\mathcal{G}_{\mathcal{M}}(\sigma)$ be the generalized eigenspace of σ in \mathcal{M} , then

$$\mathcal{G}_{\mathcal{M}}(\sigma) \simeq E_A(\mu) \oplus E_A(\mu) \oplus E_A(\mu) \oplus E_A(\mu)$$

where $E_A(\mu)$ is the eigenspace of μ and there exists a basis of $E_A(\mu) \otimes \mathbb{C}^{2n}$ such that

$$\mathcal{M}|_{\mathcal{G}_{\mathcal{M}}(\sigma)} = \begin{pmatrix} 0 & -\omega & 1 & 0 \\ \omega & 0 & 0 & 1 \\ 0 & 0 & 0 & -\omega \\ 0 & 0 & \omega & 0 \end{pmatrix} \otimes I_p.$$

The proof of this theorem shows that the basis of $\mathcal{G}_{\mathcal{M}}(\sigma)$ can be written as

$$\bigoplus_{j=1}^p \text{span}\{V_j \otimes \text{Im}(U_1), V_j \otimes \text{Re}(U_1), V_j \otimes \text{Im}(U_2), V_j \otimes \text{Re}(U_2)\} \quad (5)$$

where V_1, \dots, V_p is a basis for $E_A(\mu)$.

We remark that the cell dimension needed to have purely imaginary eigenvalues in $1 : -1$ resonance in M_μ must be at least 4. Note that the versal unfolding of the $1 : -1$ resonance is

$$\begin{pmatrix} 0 & -\omega & 1 & 0 \\ \omega & 0 & 0 & 1 \\ \lambda & 0 & 0 & -\omega \\ 0 & \lambda & \omega & 0 \end{pmatrix} \otimes I_p.$$

4. The Hamiltonian-Hopf Theorem

The bifurcation of periodic solutions occurring at a $1 : -1$ resonance in generic systems of Hamiltonian equations is described by the Hamiltonian Hopf Theorem [16]. In the context of a Hamiltonian network of coupled Hamiltonian cells, Proposition 3.1 provides the linear structure at the equilibrium solution from which we can establish the bifurcation result. However, we must identify the subspaces where the $1 : -1$ resonance can be applied and for which synchronization between some cells is achieved.

For a graph G with vertex set $V(G) = N$, the phase space P has dimension $2nN$ and we write the coordinates as $x = (x_1, \dots, x_N)$ where x_j is an element of cell j . Here, we define concepts introduced in Stewart *et al* [13] about the subspace structure of coupled cell systems. We denote by \bowtie an equivalence relation on G and define a *polydiagonal subspace* associated with \bowtie as

$$\Delta_{\bowtie} = \{x \in P \mid x_i = x_j \text{ whenever } i \bowtie j, \forall i, j \in \{1, \dots, N\}\}.$$

The subspace Δ_{\bowtie} is flow-invariant for all admissible vector fields for the network structure given by G if \bowtie is a “balanced” equivalence relation. The exact definition of balanced equivalence relation requires a lengthy discussion and we refer the reader to [13] for all the details. For homogeneous and regular networks, it can be described as follows. For an equivalence relation \bowtie , all cells in the same equivalence class are given the same colour, different from the other equivalence classes. An equivalence relation \bowtie for a homogeneous regular coupled cell system is said to be *balanced* if for every cells c, d in the vertex set V of G such that $c \bowtie d$, then cells c and d receive the same number of inputs from cells of the same colour. The balanced equivalence relation concept is therefore crucial to determine subspaces for which the dynamics on the coupled cell system is flow-invariant. We say that Δ_{\bowtie} is a *synchrony subspace* if \bowtie is balanced. In our context, we not only need the flow-invariance of the synchrony subspace, but we must make sure that the Hamiltonian structure is preserved too. This is proved in [3]. A synchrony subspace Δ is also a symplectic subspace and if H is the Hamiltonian function for the vector field on P , then $H|_\Delta$ is the Hamiltonian function for the vector field restricted to Δ .

A Hamiltonian in \mathbb{R}^4 with $1 : -1$ resonance as studied by van der Meer [16] has a normal form given by the

Hamiltonian function $H(x, y) = S + N + \mu P + aP^2$ where $S = x_1y_2 - x_2y_1$, $N = \frac{1}{2}(x_1^2 + x_2^2)$, $P = \frac{1}{2}(y_1^2 + y_2^2)$. The sign of a determines the appearance of the periodic solutions according to two different scenarios as described in [16, 1]. Write system (1) as $\dot{x} = F(x, \lambda)$. Suppose that $dF(0, 0)$ has $\pm i\omega$ eigenvalues in $1 : -1$ resonance and E is its generalized eigenspace, the following result was established.

Theorem 4.1 (Chan *et al* [3]) *Consider a 1-parameter family of Hamiltonian networks of coupled Hamiltonian cells with an equilibrium solution at the origin. Suppose the linearization at the origin has a $1 : -1$ resonance with generalized eigenspace E . Let Δ be a synchrony subspace such that $\dim(\Delta \cap E) = 4$. Let a_Δ be the coefficient of the normal form of H_0 on $\Delta \cap E$. Then, provided $a_\Delta \neq 0$, the same two scenarios occur as for the ordinary Hamiltonian Hopf bifurcation theorem.*

We illustrate Theorem 4.1 using the network of Example 2.2.

Example 4.2 *Balanced equivalence relations are obtained by coloring the cells in different colors. Note that each cell of the same color, receives the same number of inputs from a given category of color of cells. See Figure 3 for the equivalence relation leading to the synchrony subspace Δ_2 below. For instance, here, each black cell receives two inputs from white cells and two inputs from dark grey cells. We obtain synchrony sub-*

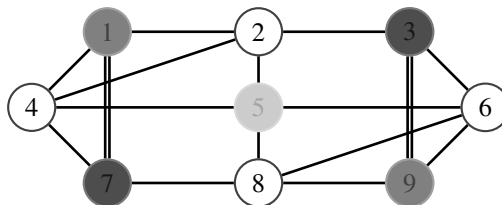


Figure 3: The Δ_1 synchrony subspace in Γ_1 .

spaces $\Delta_1 = \{(a, b, c, b, d, b, c, b, a) \mid a, b, c, d \in \mathbb{R}^k\}$, $\Delta_2 = \{(a, b, a, c, d, c, a, b, a) \mid a, b, c, d \in \mathbb{R}^k\}$. As described in [3], if $\sigma = i\omega$ occurs for the matrix M_μ where μ is one of the double eigenvalues of A , then let U_1 be the eigenvector of σ and U_2 a generalized eigenvector and let V_1, V_2 be a basis of a double eigenvalue -2 of A . Using the formula (5) one can see that $\mathcal{G}_M(\mu) \subset \Delta_1$ because $V_1, V_2 \in \Delta_1$ as can be easily verified. Thus, the periodic orbit bifurcating has the synchronization pattern of Δ_1 .

5. The Lyapunov Centre Theorem

We conclude this short article by considering an extension of the Lyapunov Centre Theorem to the case of Hamiltonian networks. This result is not found in [3]. The Lyapunov centre theorem describes the existence of a family of periodic orbits in the neighborhood of an equilibrium

solution with a pair of nonresonant purely imaginary eigenvalues [11] such that the frequency of the family of orbits converges to the frequency given by the purely imaginary eigenvalue. This result has been extended to the equivariant [8] and reversible-equivariant setting [2]. Let (1) be written as $\dot{x} = F(x)$ and assume $F(0) = 0$. Recall from [11] that eigenvalues λ of Hamiltonian matrices come in quadruplets $\lambda, \bar{\lambda}, -\lambda, -\bar{\lambda}$. We have the following result.

Theorem 5.1 *Suppose that $dF(0)$ has a pair of purely imaginary eigenvalues $\lambda_1 = i\omega$ in \mathcal{M}_μ such that all other eigenvalues λ (except $\bar{\lambda}_1$) are nonresonant; that is, $\lambda \neq k\lambda_1$ for any integer k . Let E be the eigenspace of $\lambda_1, \bar{\lambda}_1$ and Δ be a synchrony subspace such that $\dim(\Delta \cap E) = 2$. Then, there exists a 1-parameter family of periodic orbits γ_ϵ for $\epsilon \in [0, \epsilon_0]$ in $\Delta \cap E$ forming a smooth two-dimensional manifold of periodic orbits containing the equilibrium $x = 0$ and such that the period of γ_ϵ converges to $2\pi/\omega$ as $\epsilon \rightarrow 0$.*

Proof: The assumptions of the theorem guarantee that the Hamiltonian subsystem in the synchrony subspace Δ satisfies the assumptions of the standard Lyapunov Centre Theorem [11]. The conclusion follows immediately. ■

Example 5.2 *Suppose $\lambda_1 = i\omega \in \mathcal{M}_{-1}$ with complex eigenvector U_1 . The -1 eigenvalue of A in Example 2.2 has eigenvalue $W_1 = (0, 1, 0, -1, 0, -1, 0, 1, 0) \in \Delta_2$. Then, $\text{span}\{W_1 \otimes \text{Im}(U_1), W_1 \otimes \text{Re}(U_1)\} \subset \Delta_2$ and the periodic orbits have the synchronization pattern given in Figure 4.*

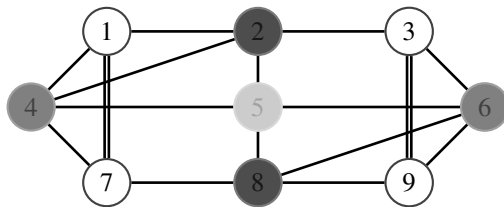


Figure 4: The Δ_2 synchrony subspace in Γ_1 .

Acknowledgments

PLB acknowledges financial support by NSERC Canada (Discovery Grant). A.P. and B.S.C. were supported by the Complex Dynamics and Systems Program of the Army Research Office, supervised Dr. Samuel Stanton, grant W911NF-07-R-003-4. We thank the reviewers for noticing mistakes and for suggestions to improve the paper.

References

[1] P-L. Buono, F. Laurent-Polz and J. Montaldi, “Symmetric Hamiltonian bifurcations,” In Geometric mechanics and symmetry, J. Montaldi, LMS Lecture Note Ser., vol 306, pp. 357–402 Cambridge Univ. Press, Cambridge, 2005

[2] C. Buzzi and J.S.W. Lamb, “Reversible Hamiltonian Liapunov center theorem,” *Discrete and Continuous Dynamical Systems B*, vol. 5, pp. 5166, 2005

[3] B. Chan, P-L. Buono and A. Palacios. “Topology and bifurcations in Hamiltonian coupled cell systems,” *Dynamical Systems: an international journal*. vol 32, pp. 23–45, 2016.

[4] M. Golubitsky, I. Stewart and A. Török, “Patterns of synchrony in coupled cell networks with multiple arrows,” *SIAM J. Appl. Dyn. Syst.*, vol 4, pp. 78–100, 2005.

[5] M.C.A. Leite and M. Golubitsky, “Homogeneous three-cell networks,” *Nonlinearity*, vol 19, pp. 2313–2363, 2006.

[6] M. Manoel and M. Roberts, “Gradient systems on coupled cell networks,” *Nonlinearity*, vol 28, pp. 3487–3509, 2015.

[7] J. Meiss, “Differential dynamical systems,” vol 14, 2007, SIAM.

[8] J. A. Montaldi, R. M. Roberts, and I. N. Stewart, “Periodic solutions near equilibria of symmetric Hamiltonian systems,” *Philosophical Transactions of the Royal Society of London A*, vol. 325, pp. 237–293, 1988.

[9] B. Rink and J. Sanders, “Coupled Cell Networks and their Hidden Symmetries,” *SIAM J. Math. Anal.*, vol 46, pp. 1577-1609, 2013.

[10] B. Rink and J. Sanders, “Amplified Hopf bifurcations in feed-forward networks,” *SIAM J. Appl. Dyn. Syst.*, vol 12, pp. 1135–1157, 2013.

[11] K.R. Meyer, G.R. Hall and Offin, D.C. Offin, “Introduction to Hamiltonian Dynamical Systems and the N-Body Problem,” 2009, Springer-Verlag, New York.

[12] T. Elmhirst and M. Golubitsky. “Nilpotent Hopf bifurcations in coupled cell systems,” *SIAM J. Appl. Dynam. Syst.*, vol 5, pp. 205-251, 2006.

[13] I. Stewart, M. Golubitsky and M. Pivato, “Symmetry Groupoids and Patterns of Synchrony in Coupled Cell Networks,” *SIAM J. Appl. Dynam. Syst.*, vol 2, pp. 609–646, 2003.

[14] M.A.D Aguiar and A.P.S. Dias, “Regular Synchrony Lattices for Product Coupled Cell Networks,” *Chaos*, vol 25, pp. 013108, 2015.

[15] M. Golubitsky and R. Lauterbach, “Bifurcations from synchrony in homogeneous networks: linear theory,” *SIAM J. Appl. Dynam. Syst.*, vol 8, pp. 40–75, 2009.

[16] J. van der Meer, *The Hamiltonian Hopf bifurcation*, Springer, New-York, 1985.