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New wave phenomena in discrete nonlinear systems

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We are concerned with breather and rogue wave solutions to a semi-discrete complex short pulse (CSP) equation. By constructing a generalized Darboux transformation and bilinear equations, the multi-breather and higher order rogue wave solutions are derived and analyzes.

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I. INTRODUCTION

The study of the nonlinear Schrödinger (NLS) equation and the coupled nonlinear Schrödinger (CNLS) equation lies at the forefront of applied mathematics and mathematical physics since it has been recognized as generic models for describing the evolution of slowly varying wave packets in general nonlinear wave system [1]. Recently, as an analogue of the nonlinear Schrödinger (NLS) equation in ultra-short pulse regime, the complex short pulse (CSP) equation

$$q_{XT} + q + \frac{1}{2}\sigma(|q|^2 q_X)_X = 0\,, \tag{1}$$

was proposed by one of the authors [2]. It can be viewed as an analogue of the nonlinear Schrödinger (NLS) equation in ultra-short pulse regime. For the focusing CSP equation ($\sigma = 1$), its multi-bright solition solution has been found in pfaffian form in [2] and in determinant form in [3] by combing Hirota's bilinear method and the Kadomtsev–Petviashvili (KP) hierarchy reduction method. In addition to above multi-bright soliton solution, the multi-breather and the higher order rogue wave solutions are constructed via Darboux tranformation method [4]. For the defocusing CSP equation ($\sigma = -1$), its multi-dark soliton solution is constructed by the KP hierarchy reduction method [6] and generalized Darboux transformation method [5], respectively. In [3, 6], the geometric formulation of the CCD equation and a geometric interpretation for the hodograph transformation was given for the focusing and defocusing CSP equation, respectively.

In the present work, as an analogue to the Ablowitz-Ladik lattice [7], we consider a semi-discrete analogue of above complex short pulse equation

$$q_{n+1,t} - q_{n,t} = \frac{a}{2}\rho_n(q_{n+1} + q_n), \tag{2}$$

$$\rho_{n,t} = -\frac{\sigma}{2a}(|q_{n+1}|^2 - |q_n|^2)\,, \tag{3}$$

where $\rho_n = (X_{n+1} - X_n)/a$, or $X_n = X_0 + a\sum_1^{n-1}\rho_n$. It is integrable since it possesses a Lax pair in the form

$$\Psi_{n+1} = U_n\Psi_n\,, \tag{4}$$

$$\Psi_{n,t} = V_n\Psi_n\,, \tag{5}$$

where

$$U_n = \begin{bmatrix} 1 - \frac{\mathrm{i}a\rho_n}{\lambda} & -\sigma\frac{q_{n+1}^* - q_n^*}{\lambda} \\ \frac{q_{n+1}-q_n}{\lambda} & 1 + \frac{\mathrm{i}a\rho_n}{\lambda} \end{bmatrix},\ V_n = \frac{\mathrm{i}}{4}\lambda\sigma_3 + \frac{\mathrm{i}}{2}Q,\ Q = \begin{bmatrix} 0 & \sigma q_n^* \\ q_n & 0 \end{bmatrix}.$$

In this paper, we are concerned with the breather and rogue wave solutions to a semi-discrete CSP equation (2) by Darboux transformation. Based on the Darboux transformation, we will firstly construct one-breather solution and multi-breather solution. Then, we can also construct first-order and higher order rogue wave (RW) solutions. The property of the 1st order breather and RW solution is analysed.

II. BREATHER AND ROGUE WAVE SOLUTIONS TO THE SEMI-DISCRRETE CSP EQUATION

Based on the Lax pair of the semi-discrete CSP equation (5), we give the Darboux transformation by the following proposition

Proposition 1 *The Darboux matrix*

$$T_n = I + \frac{\lambda_1^* - \lambda_1}{\lambda - \lambda_1^*} P_n,\ P_n = \frac{|y_{1,n}\rangle\langle y_{1,n}|J}{\langle y_{1,n}|J|y_{1,n}\rangle},\ J = \mathrm{diag}(1, \sigma), \tag{6}$$

can convert system (5) *into a new system*

$$\begin{aligned}\Psi_{n+1}^{[1]} =& U_n(\rho_n^{[1]}, q_n^{[1]}; \lambda)\Psi_n^{[1]},\\ \Psi_{n,t}^{[1]} =& V_n(\rho_n^{[1]}, q_n^{[1]}; \lambda)\Psi_n^{[1]},\end{aligned}$$

where $|y_{1,n}\rangle = (\psi_{1,n}, \phi_{1,n})^T$ *is a special solution for system* (5) *with* $\lambda = \lambda_1$, $|y_{1,n}\rangle^\dagger = \langle y_{1,n}|$. *The Bäcklund transformations between* $(\rho_n^{[1]}, q_n^{[1]})$ *and* (ρ_n, q_n) *are given through*

$$\begin{aligned}\rho_n^{[1]} =& \rho_n - 2\ln_t\left(\frac{E(\langle y_{1,n}|J|y_{1,n}\rangle)}{\langle y_{1,n}|J|y_{1,n}\rangle}\right),\\ q_n^{[1]} =& q_n + \frac{(\lambda_1^* - \lambda_1)\psi_{1,n}^*\phi_{1,n}}{\langle y_{1,n}|J|y_{1,n}\rangle},\\ |q_n^{[1]}|^2 =& |q_n|^2 + 4\sigma\ln_{tt}\left(\frac{\langle y_{1,n}|J|y_{1,n}\rangle}{\lambda_1^* - \lambda_1}\right),\end{aligned} \tag{7}$$

and the symbol E *denotes the shift operator* $n \to n+1$.

Assume that we have N different solutions $|y_{i,n}\rangle = (\psi_{i,n}, \phi_{i,n})^T$ at $\lambda = \lambda_i$ $(i = 1, 2, \cdots, N)$, then we can construct the N-fold DT.

A. Single breather solutions and multi-breather solution

The breather solution and multi-breather solution can be constructed from the seed solution–plane wave solution

$$\rho_n^{[0]} = \frac{\gamma}{2},\ q_n^{[0]} = \frac{\beta}{2}\mathrm{e}^{\mathrm{i}\theta_n},\ \theta_n = bn + \frac{c}{2}t,\ c = \frac{a\gamma}{2}\frac{\sin(b)}{\cos(b) - 1}. \tag{8}$$

After some tedious calculation, multi-breather solution can be constructed is given as follows:

Proposition 2 *The multi-breather solution for semi-discrete CSP equation* (2) *can be represented as*

$$\begin{aligned}\rho_n^{[N]} =& \frac{\gamma}{2} - \frac{2}{a}\ln_t\left(\frac{\det(M_{n+1})}{\det(M_n)}\right),\\ q_n^{[N]} =& \frac{\beta}{2}\left[\frac{\det(G_n)}{\det(M_n)}\right]\mathrm{e}^{\mathrm{i}\theta_n},\\ X_n^{[N]} =& \frac{\gamma}{2}an - \frac{\beta^2}{8}t - \frac{2}{a}\ln_t\det(M_n),\ T = -t,\end{aligned} \tag{9}$$

where

$$M_n = \left(\frac{\mathrm{e}^{\theta_{m,n}^* + \theta_{k,n}}}{\eta_m^* - \eta_k} - \frac{\mathrm{e}^{\theta_{m,n}^*}}{\eta_m^* - \chi_k} - \frac{\mathrm{e}^{\theta_{k,n}}}{\chi_m^* - \eta_m} + \frac{1}{\chi_m^* - \chi_k}\right)_{1\le m,k\le N},$$

$$G_n = \left(\frac{\mathrm{e}^{\theta_{m,n}^* + \theta_{k,n}}}{\eta_m^* - \eta_k}\frac{\eta_m^* + c}{\eta_k + c} - \frac{\mathrm{e}^{\theta_{m,n}^*}}{\eta_m^* - \chi_k}\frac{\eta_m^* + c}{\chi_k + c} - \frac{\mathrm{e}^{\theta_{k,n}}}{\chi_m^* - \eta_m}\frac{\chi_m^* + c}{\eta_k + c} + \frac{1}{\chi_m^* - \chi_k}\frac{\chi_m^* + c}{\chi_k + c}\right)_{1\le m,k\le N}.$$

The single breather solution is given by

$$\begin{aligned}\rho_n^{[1]} =& \frac{\gamma}{2} - \frac{2}{a}\ln_t\left(\frac{\cosh(\theta_{1,n+1}^R)\cosh(\varphi_1^R/2) - \sin(\theta_{1,n+1}^I)\sin(\varphi_1^I/2)}{\cosh(\theta_{1,n}^R)\cosh(\varphi_1^R/2) - \sin(\theta_{1,n}^I)\sin(\varphi_1^I/2)}\right),\\ q_n^{[1]} =& \frac{\beta}{2}\left[\frac{\cosh(\theta_{1,n}^R - \mathrm{i}\varphi_1^I)\cosh(\varphi_1^R/2) + \sin(\theta_{1,n}^I + \mathrm{i}\varphi_1^R)\sin(\varphi_1^I/2)}{\cosh(\theta_{1,n}^R)\cosh(\varphi_1^R/2) - \sin(\theta_{1,n}^I)\sin(\varphi_1^I/2)}\right]\mathrm{e}^{\mathrm{i}\theta_n},\\ X_n^{[1]} =& \frac{\gamma}{2}an - \frac{\beta^2}{8}t - \frac{2}{a}\ln_t\left(\cosh(\theta_{1,n}^R)\cosh(\varphi_1^R/2) - \sin(\theta_{1,n}^I)\sin(\varphi_1^I/2)\right),\ T = -t,\end{aligned} \tag{10}$$

where

$$\begin{aligned}\theta_{1,n}^R &= \frac{\ln(g_1)}{2}n - \frac{\beta}{2}\sinh\left(\frac{\varphi_1^R}{2}\right)\sin\left(\frac{\varphi_1^I}{2}\right)t - \varphi_1^R + a_1^R,\\ \theta_{1,n}^I &= h_1 n + \frac{\beta}{2}\cosh\left(\frac{\varphi_1^R}{2}\right)\cos\left(\frac{\varphi_1^I}{2}\right)t - \varphi_1^I + a_1^I,\end{aligned}$$

and

$$\begin{aligned}g_1 &= \frac{\beta^2\cosh^2\left(\varphi_1^R/2\right)\sin^2\left(b/2 + \varphi_1^I/2\right) + \left[\beta\sinh\left(\varphi_1^R/2\right)\cos\left(b/2 + \varphi_1^I/2\right) + \frac{a\gamma}{2\sin(b/2)}\right]^2}{\beta^2\cosh^2\left(\varphi_1^R/2\right)\sin^2\left(b/2 - \varphi_1^I/2\right) + \left[\beta\sinh\left(\varphi_1^R/2\right)\cos\left(b/2 - \varphi_1^I/2\right) + \frac{a\gamma}{2\sin(b/2)}\right]^2},\\ h_1 &= \arg\left(\frac{\sin(\frac{b}{2})\left(\frac{1}{2}\mathrm{i}a\gamma - \beta\cosh\left[\frac{1}{2}(\varphi_1^R + \mathrm{i}\varphi_1^I)\right]\right) + \mathrm{i}\cos(\frac{b}{2})(\beta\sinh\left[\frac{1}{2}(\varphi_1^R + \mathrm{i}\varphi_1^I)\right] - c)}{\sin(\frac{b}{2})\left(\frac{1}{2}\mathrm{i}a\gamma + \beta\cosh\left[\frac{1}{2}(\varphi_1^R + \mathrm{i}\varphi_1^I)\right]\right) + \mathrm{i}\cos(\frac{b}{2})(\beta\sinh\left[\frac{1}{2}(\varphi_1^R + \mathrm{i}\varphi_1^I)\right] - c)}\right).\end{aligned}$$

Specially, if we choose the parameters such that $\beta = \gamma = 1$, $a = 2$, $b = \frac{\pi}{2}$, $\varphi_{1R} = 0$, $\varphi_{1I} = \arcsin(\frac{3}{5})$, $a_1 = 0$, the breather solution which is periodical in time and localized in space (the K-M breather) is illstrated in Fig. 1(a).

B. Rogue wave and high order rogue wave solution

In above subsection, we solved the linear system (5) with plane wave seed solution by the restriction $\lambda_i \neq -c + \mathrm{i}\beta$. It is naturally to ask how about $\lambda_i = -c + \mathrm{i}\beta$. Actually, we can obtain the rogue wave solution and higher order rogue wave solutions at this special point. Specially, the first order rogue wave solution is given by

$$\begin{aligned}\rho_n^{[1]} =& \frac{\gamma}{2} - \frac{2}{a}\ln_t\left(\frac{\frac{1}{4} + (Z_{n+1,R}^{[1]})^2 + (Z_{n+1,I}^{[1]} + \frac{1}{2})^2}{\frac{1}{4} + (Z_{n,R}^{[1]})^2 + (Z_{n,I}^{[1]} + \frac{1}{2})^2}\right),\\ q_n^{[1]} =& \frac{\beta}{2}\left[1 - \frac{1 - 2\mathrm{i}Z_{n,R}^{[1]}}{\frac{1}{4} + (Z_{n,R}^{[1]})^2 + (Z_{n,I}^{[1]} + \frac{1}{2})^2}\right]\mathrm{e}^{\mathrm{i}\theta},\\ X_n^{[1]} =& \frac{\gamma}{2}an - \frac{\beta^2}{8}t - \frac{2}{a}\ln_t\left(\frac{1}{4} + (Z_{n,R}^{[1]})^2 + (Z_{n,I}^{[1]} + \frac{1}{2})^2\right),\ T = -t,\end{aligned}$$

where

$$\begin{aligned}Z_{n,R}^{[1]} =& \frac{4\beta^2\sin^3(\frac{b}{2})\cos(\frac{b}{2})n}{a^2\gamma^2 + 2\beta^2\sin^2(b)},\\ Z_{n,I}^{[1]} =& \beta\left(\frac{2a\gamma\sin^2(\frac{b}{2})n}{a^2\gamma^2 + 2\beta^2\sin^2(b)} + \frac{t}{4}\right) - \frac{1}{2}.\end{aligned} \tag{11}$$

We show such a fundamental rogue wave in Fig. 1(b) with the parameters $a = 2$, $b = \frac{\pi}{2}$, $\beta = 1$, $\gamma = \frac{5}{4}$.

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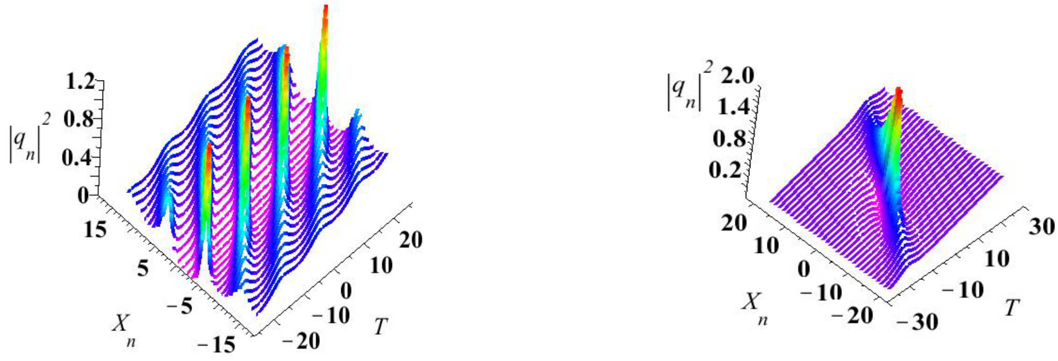

(a)K-M breather

(b)Fundamental RW

FIG. 1: (color online): Breather and Rogue waves

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