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Spatial properties and numerical solitary waves of a nonintegrable discrete nonlinear Schrödinger equation

Li-Yuan Ma†, Zuo-Nong Zhu‡ *
†Department of Applied Mathematics, Zhejiang University of Technology,
Hangzhou 310023, P. R. China
‡School of Mathematical Sciences, Shanghai Jiao Tong University,
800 Dongchuan Road, Shanghai, 200240, P. R. China
Email: mly2016@zjut.edu.cn, znzhu@sjtu.edu.cn

Abstract

In this paper, we study a nonintegrable discrete nonlinear Schrödinger (dNLS) equation with nonlinear interaction terms. By using the planar nonlinear dynamical map approach, we address the spatial properties of the nonintegrable dNLS equation. Through the constructions of exact period-1 and period-2 orbits of a planar nonlinear map which is a stationary version of the nonintegrable dNLS equation, we obtain the spatially periodic solutions of the nonintegrable dNLS equation. We also give the numerical simulations of the orbits of the planar nonlinear map and show how the nonlinear interaction terms affect those orbits. By using discrete Fourier transformation method, we obtain numerical approximations of stationary and travelling solitary wave solutions of the nonintegrable dNLS equation.

1 Introduction

As is well known, discrete nonlinear Schrödinger equation has many important applications in the various physical fields, e.g., atomic chains with on-site cubic nonlinearities, biological system, Bose-Einstein condensates, nonlinear coupled optical waveguide. Discrete soliton as its localized mode has been paid attention to by many researchers. In this paper, we will study the following nonintegrable dNLS equation with the nonlinear hopping

$$i\frac{dq_n}{dt}+(1+\mu|q_n|^2)(q_{n+1}+q_{n-1})-2q_n+\alpha q_n(\bar{q}_{n+1}q_{n-1}+q_{n+1}\bar{q}_{n-1})+\beta q_n^2(\bar{q}_{n+1}+\bar{q}_{n-1})-2\gamma|q_n|^2q_n=0, \tag{1}$$

where the free parameters $\mu, \alpha, \beta, \gamma$ are real. We will investigate the spatial properties, stationary and travelling solitary wave solutions of the nonintegrable dNLS equation and reveal the importance of the nonlinear interaction terms $\alpha q_n(\bar{q}_{n+1}q_{n-1} + q_{n+1}\bar{q}_{n-1})$ and $\beta q_n^2(\bar{q}_{n+1} + \bar{q}_{n-1})$. Several nonintegrable dNLS equations related to equation (1) have been investigated. For example, in Refs. [1, 2], the spatial dynamics including stationary and wave transmission properties of the nonintegrable dNLS equation (1) with the special case of $\alpha = \beta = 0$ were considered by utilizing a planar nonlinear dynamical map approach. Besides, the gauge equivalence, existence and stability of traveling solutions of this special case were also discussed in [2, 3]. Ablowitz and Musslimani studied a special case of equation (1) with

*Corresponding author

$\alpha = \beta = \mu = 0$. By using discrete Fourier analysis method, numerical approximations of stationary and traveling solitary wave of this special equation were discussed [4]. Existence and stability of solitary waves of this special equation can be also obtained via variational principles or the analysis of linear spectrum. Many important developments of the dNLS equation (1) with $\alpha = \beta = \mu = 0$ including ground and excited states, their construction, stability, bifurcations and mobile breathers etc were reviewed in [5].

In this paper, we focus on the spatial properties and solitary wave solutions of nonintegrable dNLS equation (1). By using the planar nonlinear dynamical map approach, we address the spatial properties of the nonintegrable dNLS equation. Through the constructions of exact period-1 and period-2 orbits of a planar nonlinear map which is a stationary version of the nonintegrable dNLS equation, we obtain the spatially periodic solutions of the nonintegrable dNLS equation. We also give the numerical simulations of the orbits of the planar nonlinear map and show how the nonlinear interaction terms affect those orbits. By using discrete Fourier transformation, we obtain numerical approximations of stationary and travelling solitary wave solutions of the nonintegrable dNLS equation and show that the nonlinear interaction terms have much more influence on the form of solitary wave.

2 Spatially periodic solutions from planar dynamical map approach

2.1 A plane map related to the stationary dNLS equation

Set $q_n(t) = r_n e^{i\theta_n + i(F-2)t}$. Eq. (1) is converted into the following equation:

$$r_{n+1}\cos(\Delta\theta_{n+1}) + r_{n-1}\cos(\Delta\theta_n) = \frac{(F + 2\gamma r_n^2)r_n - 2\alpha r_{n+1}r_n r_{n-1}\cos(\Delta\theta_{n+1} + \Delta\theta_n)}{1 + (\mu+\beta)r_n^2}, \quad (2)$$

$$r_{n+1}\sin(\Delta\theta_{n+1}) - r_{n-1}\sin(\Delta\theta_n) = 0, \quad (3)$$

where $\Delta\theta_n = \theta_n - \theta_{n-1}$. Eq.(3) implies a conserved quantity of the probability current

$$J = r_n r_{n-1}\sin(\Delta\theta_n). \quad (4)$$

Through introducing real-valued variables transformations:

$$x_n = 2r_n r_{n-1}\cos(\Delta\theta_n), \qquad y_n = 2J, \qquad z_n = r_n^2 - r_{n-1}^2, \quad (5)$$

equations (2) and (3) are equivalent to a two-dimensional real map $M_{\alpha,\beta,\gamma,\mu,F,J}$

$$M_{\alpha,\beta,\gamma,\mu,F,J} : \begin{cases} x_{n+1} = \frac{(F+\gamma(\omega_n+z_n))(\omega_n+z_n)+4\alpha J^2 - x_n\left(1+\frac{\mu+\beta}{2}(\omega_n+z_n)\right)}{1+\frac{\mu+\beta}{2}(\omega_n+z_n)+\alpha x_n}, \\ z_{n+1} = \frac{x_{n+1}^2 - x_n^2}{2(\omega_n+z_n)} - z_n, \end{cases} \quad (6)$$

where $\omega_n = \sqrt{x_n^2 + z_n^2 + 4J^2}$.

2.2 Periodic orbits of the map (6)

Case 1: $J = 0$

Considering the peculiar orbit with $z_n = 0, \forall n$, we obtain $x_{n+1} = \pm x_n, \forall n$. This means that the period-1 orbit and the period-2 orbit of the map (6) are constructed. The period-1 orbit is the fixed point of the two-dimensional real map (6),

$$x = x_0 = \frac{F-2}{\mu + \alpha + \beta - \gamma} > 0, \quad z = 0, \quad (7)$$

or

$$x = x_0 = \frac{F+2}{\mu+\beta+\gamma-\alpha} < 0, \quad z = 0. \tag{8}$$

The period-2 orbit is

$$x = x_0 = \frac{-F}{\alpha+\gamma} > 0, \quad z = 0, \tag{9}$$

or

$$x = x_0 = \frac{F}{\alpha+\gamma} < 0, \quad z = 0, \tag{10}$$

which creates period-doubling bifurcation for the map $M_{\alpha,\beta,\gamma,\mu,F,J}$ (6).

Case 2: $J \neq 0$

Introducing the scaling transformation: $x_n = 2J\tilde{x}_n$, $z_n = 2J\tilde{z}_n$, $J\gamma = \tilde{\gamma}$ and $\omega_n = 2J\tilde{\omega}_n$, we can rewrite the map $M_{\alpha,\beta,\gamma,\mu,F,J}$ as,

$$M_{\alpha,\beta,\mu,\tilde{\gamma},F,J} : \begin{cases} \tilde{x}_{n+1} = \frac{(F+2\tilde{\gamma}(\tilde{\omega}_n+\tilde{z}_n))(\tilde{\omega}_n+\tilde{z}_n)+2\alpha J-\tilde{x}_n(1+J(\mu+\beta)(\tilde{\omega}_n+\tilde{z}_n))}{1+J(\mu+\beta)(\tilde{\omega}_n+\tilde{z}_n)+2\alpha J\tilde{x}_n}, \\ \tilde{z}_{n+1} = \frac{\tilde{x}_{n+1}^2-\tilde{x}_n^2}{2(\tilde{\omega}_n+\tilde{z}_n)} - \tilde{z}_n, \end{cases} \tag{11}$$

with $\tilde{\omega}_n = \sqrt{\tilde{x}_n^2+\tilde{z}_n^2+1}$ as $J > 0$, or $\tilde{\omega}_n = -\sqrt{\tilde{x}_n^2+\tilde{z}_n^2+1}$ as $J < 0$.

Setting $\tilde{z}_n = 0, \forall n$, we get $\tilde{x}_{n+1} = \tilde{x}_n$, i.e., $\tilde{x}_n$ is the period-1 orbit, and $\tilde{x}_{n+1} = -\tilde{x}_n$, i.e. $\tilde{x}_n$ is the period-2 orbit. The period-1 orbit is determined by

$$2(\tilde{\gamma}-\alpha J)x^2 - 2x - 2J(\mu+\beta)x\omega + F\omega + 2(\alpha J+\tilde{\gamma}) = 0, \quad z = 0 \tag{12}$$

where $\omega = \pm\sqrt{1+x^2}$. Considering a special case $\alpha+\gamma = 0$ and $F = 0$, we have

$$2\alpha Jx \pm J(\mu+\beta)\sqrt{1+x^2} + 1 = 0, \quad z = 0.$$

When $\mathrm{sgn}(2\alpha Jx+1) = \mathrm{sgn}(\pm J(\mu+\beta)\sqrt{1+x^2})$ and $1 + J^2\left(4\alpha^2-(\mu+\beta)^2\right) \geq 0$, the exact period-1 orbit is given by

$$x = \tilde{x}_0 = \frac{-2\alpha \pm (\mu+\beta)\sqrt{1+J^2\left(4\alpha^2-(\mu+\beta)^2\right)}}{J\left(4\alpha^2-(\mu+\beta)^2\right)}, \quad z = 0 \tag{13}$$

When $\frac{-F}{J(\alpha+\gamma)} > 0$, and $|F| > 2|J(\alpha+\gamma)|$, the period-2 orbits are

$$x = \tilde{x}_0 = \sqrt{\frac{F^2}{4J^2(\alpha+\gamma)^2} - 1}, \quad z = 0. \tag{14}$$

or

$$x = \tilde{x}_0 = -\sqrt{\frac{F^2}{4J^2(\alpha+\gamma)^2} - 1}, \quad z = 0. \tag{15}$$

2.3 Exact spatially periodic solutions of nonintegrable dNLS equation

We give the spatially periodic solutions of the nonintegrable dNLS equation (1). One can see that the periodicity of the orbits of the plane map does not coincide with the space periodicity of the solution. This is an interesting phenomenon for the nonintegrable dNLS equation (1).

For the probability current $J = 0$, the period-1 orbit (7) yields a period-1 solution

$$q_n(t) = \sqrt{\frac{x_0}{2}} e^{i((F-2)t+\theta_0)} \tag{16}$$

of the nonintegrable dNLS equation, where θ_0 is the argument of φ_0. However, for another period-1 orbit (8), its corresponding solution is a period-2 solution,

$$\begin{cases} q_0(t) = \sqrt{\frac{-x_0}{2}} e^{i[(F-2)t+\theta_0]}, \\ q_1(t) = -\sqrt{\frac{-x_0}{2}} e^{i[(F-2)t+\theta_0]}, \\ q_{n+2}(t) = q_n(t) \quad \forall n, \end{cases} \tag{17}$$

where $x_0 = \frac{F+2}{\mu-\alpha+\beta+\gamma} < 0$.
The period-2 orbit (9) yields the following period-4 solution to the nonintegrable dNLS equation

$$\begin{cases} q_0(t) = q_3(t) = \sqrt{\frac{x_0}{2}} e^{i[(F-2)t+\theta_0]}, \\ q_1(t) = q_2(t) = -\sqrt{\frac{x_0}{2}} e^{i[(F+2)t+\theta_0]}, \\ q_{n+4}(t) = q_n(t) \quad \forall n, \end{cases} \tag{18}$$

where $x_0 = \frac{-F}{\alpha+\gamma} > 0$. But, another period-2 orbit (12) can not yield the corresponding solution to nonintegrable dNLS equation.

For $J \neq 0$, when period-1 orbit $x = \tilde{x}_0 > 0$, the generating solution

$$q_n(t) = re^{i\left((F-2)t+\theta_0+n\arcsin(J/r^2)\right)} \tag{19}$$

to the nonintegrable dNLS equation is given by the period-1 orbit (12), where r admits the constraint condition: $J^2(1+\tilde{x}_0^2) = r^4$. It is not period in general. But when $\arcsin\frac{J}{r^2} = \frac{2\pi}{m}, \quad \forall m \in Z^+$ and $m > 3$, it is a period-m solution. For the period-2 orbit (15), its yielding solution is a period-4 solution

$$q_n(t) = re^{i\left((F-2)t-[\frac{n}{2}]\pi+\frac{1+(-1)^{n+1}}{2}\arcsin\frac{J}{r^2}\right)}. \tag{20}$$

Like the case of $J = 0$, in the case of $J \neq 0$, another period-2 orbit can not yield the corresponding solution to nonintegrable dNLS equation.
We remark here that the stability of orbits of the plane map, the numerical simulations for the orbit of stationary dNLS equation (11) in the general case of $z_n \neq 0$, numerical approximations of localized stationary and traveling solitary wave solutions of nonintegrable dNLS equation (1) have been given (see Ref. [6]).

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