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Phase Reduction Theory for Strongly Coupled Limit-Cycle Oscillators

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Abstract—Unlike the conventional phase reduction method limited to weakly perturbed oscillators, a generalized phase reduction method [W. Kurebayashi et al., Phys. Rev. Lett., 2013] has enabled the reduction of a strongly perturbed limit-cycle oscillator to a one-dimensional phase model. In this study, by applying the generalized phase reduction method, we derive a new phase model that enables theoretical analysis of strongly coupled oscillators. Since this method can explicitly take into account the deformation of orbits due to strong perturbations, our model can describe nontrivial phase dynamics of strongly coupled oscillators on deformed manifolds.

1. Formulation and phase model

We consider two coupled limit-cycle oscillators:

$$\dot{\boldsymbol{X}}^{(j)}(t) = \boldsymbol{F}^{(j)}(\boldsymbol{X}^{(j)}(t), \boldsymbol{I}^{(j)}(\epsilon t)), \tag{1}$$

$$\boldsymbol{I}^{(j)}(\epsilon t) = \boldsymbol{G}^{(j)}(\boldsymbol{X}^{(1)}(t), \boldsymbol{X}^{(2)}(t)), \tag{2}$$

for $j = 1, 2$, where $\boldsymbol{X}^{(j)}(t) \in \mathbb{R}^n$ is a state variable of the j-th oscillator, $\boldsymbol{F}^{(j)}(\boldsymbol{X}, \boldsymbol{I}) \in \mathbb{R}^n$ is a vector field that represents the dynamics of the jth oscillator and has a stable limit-cycle orbit with period $T^{(j)}(\boldsymbol{I})$ and frequency $\omega^{(j)}(\boldsymbol{I}) = 2\pi/T(\boldsymbol{I})$ for each fixed $\boldsymbol{I}$, $\boldsymbol{I}^{(j)}(\epsilon t) \in \mathbb{R}^m$ represents an input from the other oscillator to the j-th oscillator defined by the function $\boldsymbol{G}^{(j)}(\boldsymbol{X}^{(1)}, \boldsymbol{X}^{(2)}) \in \mathbb{R}^m$, and ϵ is a small parameter that represents the relative time scale of the input $\boldsymbol{I}^{(j)}(\epsilon t)$ as compared to amplitude relaxation, which can be assumed to be $O(1)$ without loss of generality. We define the cylinder of limit cycles as $\boldsymbol{X}_0^{(j)}(\theta, \boldsymbol{I})$ with a phase parameter θ so that $\boldsymbol{X}^{(j)}(t) = \boldsymbol{X}_0^{(j)}(\omega(\boldsymbol{I})t, \boldsymbol{I})$ for each fixed $\boldsymbol{I}$.

By using the generalized phase reduction method [1], we can define the phase variable $\theta^{(j)}(t)$ of the j-th oscillator and its dynamics is described by

$$\begin{aligned}\dot{\theta}^{(j)} &= \omega^{(j)}(\boldsymbol{I}^{(j)}(\epsilon t)) + \epsilon \boldsymbol{\xi}^{(j)}(\theta^{(j)}, \boldsymbol{I}^{(j)}(\epsilon t)) \cdot \dot{\boldsymbol{I}}^{(j)}(\epsilon t) \\ &\quad + O(\epsilon^2),\end{aligned} \tag{3}$$

where $\dot{\boldsymbol{I}}^{(j)}(\epsilon t)$ denotes $d\boldsymbol{I}^{(j)}(\epsilon t)/d(\epsilon t)$, and $\boldsymbol{\xi}^{(j)}(\theta, \boldsymbol{I}) \in \mathbb{R}^m$ is a sensitivity function that characterizes the response property of the j-th oscillator to the applied input $\boldsymbol{I}^{(j)}(\epsilon t)$.

2. Phase model for strongly coupled oscillators

Equation (3) cannot be directly used for synchronization analysis of coupled oscillators, because Eq. (3) is not closed for the phase variables $\theta^{(j)}(t)$. In this section, we will derive a closed phase model by projecting the phase dynamics onto a two-dimensional manifold consisting of deformed limit cycles of the two coupled oscillators.

We assume that the following equations have a solution for given $\theta^{(1)}, \theta^{(2)}$:

$$\boldsymbol{G}^{(1)}(\boldsymbol{X}_0^{(1)}(\theta^{(1)}, \boldsymbol{I}^{(1)}), \boldsymbol{X}_0^{(2)}(\theta^{(2)}, \boldsymbol{I}^{(2)})) = \boldsymbol{I}^{(1)}, \tag{4}$$

$$\boldsymbol{G}^{(2)}(\boldsymbol{X}_0^{(1)}(\theta^{(1)}, \boldsymbol{I}^{(1)}), \boldsymbol{X}_0^{(2)}(\theta^{(2)}, \boldsymbol{I}^{(2)})) = \boldsymbol{I}^{(2)}, \tag{5}$$

and denote their solution by $\boldsymbol{I}^{(1)} = \boldsymbol{I}_0^{(1)}(\boldsymbol{\theta})$ and $\boldsymbol{I}^{(2)} = \boldsymbol{I}_0^{(2)}(\boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is introduced as follows:

$$\boldsymbol{\theta} = [\theta^{(1)}, \theta^{(2)}]^\top. \tag{6}$$

By using the notations $\boldsymbol{I}_0^{(1)}(\boldsymbol{\theta})$ and $\boldsymbol{I}^{(2)}(\boldsymbol{\theta})$, we difine

$$\boldsymbol{\omega}(\boldsymbol{\theta}) = [\omega^{(1)}(\boldsymbol{I}_0^{(1)}(\boldsymbol{\theta})), \omega^{(2)}(\boldsymbol{I}_0^{(2)}(\boldsymbol{\theta}))]^\top, \tag{7}$$

and

$$\Xi(\boldsymbol{\theta}) = \begin{pmatrix} \Xi_{11}(\boldsymbol{\theta}) & \Xi_{12}(\boldsymbol{\theta}) \\ \Xi_{21}(\boldsymbol{\theta}) & \Xi_{22}(\boldsymbol{\theta}) \end{pmatrix}, \tag{8}$$

where the (i, j)-th element of $\Xi(\boldsymbol{\theta})$ is given by

$$\Xi_{ij}(\boldsymbol{\theta}) = \boldsymbol{\xi}^{(i)}(\theta, \boldsymbol{I}^{(i)}(\boldsymbol{\theta})) \cdot \frac{\partial \boldsymbol{I}_0^{(i)}(\boldsymbol{\theta})}{\partial \theta^{(j)}}. \tag{9}$$

Under the assumption that the matrix $\Xi(\boldsymbol{\theta})$ is not singular, we can derive the following phase model:

$$\dot{\boldsymbol{\theta}}(t) = \Xi^{-1}(\boldsymbol{\theta}(t))\boldsymbol{\omega}(\boldsymbol{\theta}(t)), \tag{10}$$

In this study, we propose Eq. (10) as a new phase model for strongly coupled oscillators.

References

[1] W. Kurebayashi, S. Shirasaka, and H. Nakao, Phys. Rev. Lett. 111, 214101 (2013).