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Stability analysis of partial amplitude death on five delay-coupled Stuart-Landau oscillators

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Abstract—The present study investigates partial amplitude death on five delay-coupled Stuart-Landau oscillators. It was known that partial amplitude death occurs in coupled oscillators if the oscillators have different frequency each other or are stabilized at steady state before coupling. The linear stability analysis reveals that we can induce partial amplitude death in the coupled oscillators even if the above conditions are not satisfied. The analytical results are confirmed by numerical simulations.

1. Introduction

The various phenomena in coupled oscillators have been intensively investigated in nonlinear science [1]. The weak coupling, which effects the phase of the oscillators, induces in-phase synchronization, anti-phase synchronization, splay state, and chimera state [2, 3]. The strong coupling, which effects the amplitude of the oscillators, can induce the suppressing of oscillation phenomena: amplitude death [4] and oscillation death [5]. Amplitude death is a stabilization of homogeneous steady state while oscillation death is a stabilization of heterogeneous steady state [5].

Partial amplitude death is a phenomenon in which some of the oscillators are suppressed but the others continue to oscillate [6, 7]. Atay reported that partial amplitude death can be induced in the delay-coupled oscillators if oscillators have different frequency each other. Poel showed that partial amplitude death occurs in coupled-identical Stuart-Landau oscillators if the steady state of the oscillators are stabilized before coupling (i.e., independent oscillators are not oscillating). To our knowledge, as shown the above, the necessary conditions for inducing partial amplitude death are the followings: the oscillators are not identical or independent oscillators are stabilized at steady state.

The present study investigates partial amplitude death in five delay-coupled Stuart-Landau oscillators. Linear stability analysis reveals that partial amplitude death can be observed in the coupled oscillators even if the above conditions are not satisfied. These results are confirmed via numerical simulations.

2. Mathematical model

The present study considers coupled identical Stuart-Landau oscillators on a graph illustrated in Fig. 1(a),

$$\dot{z}_i(t) = f(z_i(t)) + u_i(t), \qquad i \in \{1, \dots, 5\}, \tag{1}$$

where $z_i(t) \in \mathbb{C}$ denotes the state variable of the i-th oscillator. $u_i(t)$ is the input signal. The nonlinear function $f(\cdot)$ is given by,

$$f(z) = (\mu + j\omega - |z|^2)z, \tag{2}$$

where $j := \sqrt{-1}$. The parameters μ and $\omega > 0$ respectively denote the instability and the natural frequency of the equilibrium point $z^* = 0$: for $\mu > 0$, the independent oscillator has the stable limit cycle solution $z(t) = \sqrt{\mu}e^{j\omega t}$; for $\mu < 0$, the independent oscillator is stabilized at the origin (i.e., not oscillating). The oscillators are coupled with the coupling strength k and the connection delay τ as follows:

$$u_i(t) = k\left(\sum_{l=1}^{5} a_{il} z_l(t-\tau) - z_i(t)\right), \tag{3}$$

where $a_{il} := \{\boldsymbol{A}\}_{i,l}$ is the (i, l) elements of the normalized adjacency matrix $\boldsymbol{A}$ of the graph illustrated in Fig. 1(a),

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 & 0 \\ 1/4 & 1/4 & 0 & 1/4 & 1/4 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 & 0 \end{pmatrix}. \tag{4}$$

The self-delayed feedback is not considered in the present study.

3. Linear stability analysis

This section analyzes the local stability of partial amplitude death in coupled oscillators (1) (2) (3). First, solutions of partial amplitude death are estimated by using the method in [7]. Second, we show that the local stability of these solutions is equivalent to that of time-invariant linear system.

3.1. Patterns of partial amplitude death

The solutions of partial amplitude death will be derived from the adjacency matrix $\boldsymbol{A}$. The eigenvectors and the eigenvalues of $\boldsymbol{A}$ are given by

$$\begin{aligned}
\boldsymbol{v}^{(1)} &= (1,1,1,1,1)^T, & \eta_1 &= 1,\\
\boldsymbol{v}^{(2)} &= (1,-1,0,1,-1)^T, & \eta_2 &= -2/3,\\
\boldsymbol{v}^{(3)} &= (0,1,0,0,-1)^T, & \eta_3 &= 0,\\
\boldsymbol{v}^{(4)} &= (1,0,0,-1,0)^T, & \eta_4 &= 0,\\
\boldsymbol{v}^{(5)} &= (1,1,-3,1,1)^T, & \eta_5 &= -1/3,
\end{aligned} \tag{5}$$

where these eigenvalues and eigenvectors satisfy

$$\boldsymbol{A}\boldsymbol{v}^{(i)} = \eta_i \boldsymbol{v}^{(i)}. \tag{6}$$

We will focus on the eigenvectors $\boldsymbol{v}^{(1)}$, $\boldsymbol{v}^{(2)}$, $\boldsymbol{v}^{(3)}$, and $\boldsymbol{v}^{(4)}$ whose elements are ± 1 or 0. Note that, for a constant value $v \in \{\pm 1, 0\}$, Stuart-Landau oscillator (2) satisfies

$$f(vz) = v f(z). \tag{7}$$

Here, we assume that Eq. (1) has the solution,

$$z_i(t) = v_i z_\eta(t), \tag{8}$$

where v_i is the i-th element of the eigenvectors $\boldsymbol{v}^{(1)}$, $\boldsymbol{v}^{(2)}$, $\boldsymbol{v}^{(3)}$, and $\boldsymbol{v}^{(4)}$, whose elements are ± 1 or 0. Furthermore, substituting it into Eq. (1), we obtain

$$v_i \dot{z}_\eta(t) = v_i f(z_\eta(t)) - k v_i z_\eta(t) + k\eta v_i z_\eta(t-\tau), \tag{9}$$

where η is the eigenvalue corresponding to the eigenvector. For $v_i = 0$, we can easily confirm that Eq. (9) holds. For $v_i \neq 0$ (i.e., $v_i = \pm 1$), Eq. (9) can be rewritten as

$$\dot{z}_\eta(t) = f(z_\eta(t)) - k z_\eta(t) + k\eta z_\eta(t-\tau). \tag{10}$$

We notice that Eq. (10) is the dynamics of the Stuart Landau oscillator with delayed feedback control. It is known that Eq. (10) has periodic solutions (see appendix A). We define the periodic solution of Eq. (10) as $z_\eta(t) = ae^{j\lambda t}$. From Eq. (8), each oscillator in the coupled oscillators (1) has the solution $z_i(t) = 0$ or $z_i(t) = \pm z_\eta(t)$ corresponding to $v_i = 0$ and $v_i = \pm 1$. For instance, for $\boldsymbol{v}^{(2)}$ in Eq. (5), the 1-st and the 4-th oscillators have the limit cycle solution $z_1(t) = z_4(t) = z_\eta(t)$, and the 2-nd and the 5-th oscillators have the solution $z_2(t) = z_5(t) = -z_\eta(t)$, and the 3-rd oscillator has the steady state solution $z_3(t) = 0$.

Figure 1(b) illustrates the periodic solutions corresponding to $v_i = 0$ and $v_i = \pm 1$ in the phase plane. The periodic solutions corresponding $v_i = 1$ and $v_i = -1$ synchronize in anti-phase. Figures 2(a)-(d) show all the patterns of coupled oscillators estimated by the eigenvectors $\boldsymbol{v}^{(1)}$, $\boldsymbol{v}^{(2)}$, $\boldsymbol{v}^{(3)}$, and $\boldsymbol{v}^{(3)} + \boldsymbol{v}^{(4)}$, respectively. Pattern (b) corresponds to the eigenvector $\boldsymbol{v}^{(2)}$, that is, only the 3-rd oscillator stays at the origin and the others oscillates.

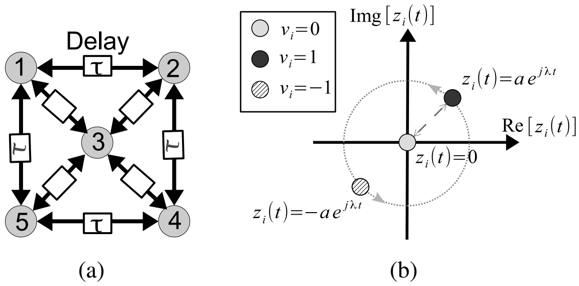


Figure 1: Delay-coupled five Stuart Landau oscillators. (a) sketch of coupled oscillators. (b) periodic solution in phase plane corresponding to the eigenvectors of adjacency matrix $\boldsymbol{A}$.

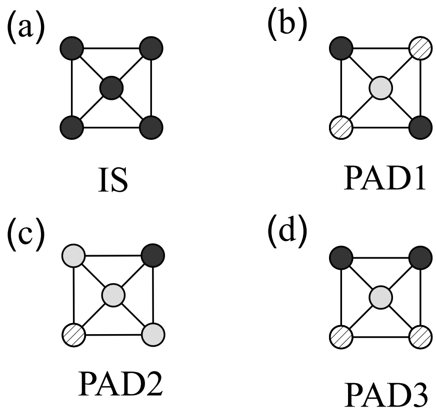


Figure 2: Sketch of patterns estimated from the eigenvectors of adjacency matrix $\boldsymbol{A}$ for the oscillators in Fig. 1(a). Surface color of each node corresponds to the solution illustrated in Fig. 1(b). Each patterns corresponds to the following eigenvectors: (a) $\boldsymbol{v}^{(1)}$, (b) $\boldsymbol{v}^{(2)}$, (c) $\boldsymbol{v}^{(3)}$, (d) $\boldsymbol{v}^{(3)} + \boldsymbol{v}^{(4)}$.

3.2. Stability of partial amplitude death

This section analyzes the local stability of partial amplitude death illustrated in Figs. 2(b)-(d). Substituting $z_i(t) = x_i(t) + jy_i(t)$ into Eq. (1), we obtain its real and imaginary parts:

$$\begin{aligned}
\dot{x}_i(t) &= \left(\mu - x_i(t)^2 - y_i(t)^2\right) x_i(t) - \omega y_i(t)\\
&\qquad -k x_i(t) + k \sum_{l=1}^{5} a_{il} x_l(t-\tau),
\end{aligned} \tag{11}$$

$$\begin{aligned}
\dot{y}_i(t) &= \left(\mu - x_i(t)^2 - y_i(t)^2\right) y_i(t) + \omega x_i(t)\\
&\qquad -k y_i(t) + k \sum_{l=1}^{5} a_{il} y_l(t-\tau).
\end{aligned} \tag{12}$$

We define the perturbation from partial amplitude death solution $\delta x_i(t) := x_i(t) - v_i \mathrm{Re}[z_\eta(t)]$ and $\delta y_i(t) := y_i(t) - v_i \mathrm{Im}[z_\eta(t)]$. Then, the dynamics around the solution is given by

$$\dot{\boldsymbol{X}}_i(t) = \left\{(\mu - k - 2v_i^2 a^2)\boldsymbol{I}_2 + \omega\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\}\boldsymbol{X}_i(t)$$
$$-v_i^2 a^2 \boldsymbol{J}_1(t)\boldsymbol{X}_i(t) + k\sum_{l=1}^{5} a_{il}\boldsymbol{X}_l(t-\tau), \quad (13)$$

where $\boldsymbol{X}_i := [\delta x_i(t), \delta y_i(t)]^T$ and,

$$\boldsymbol{J}_1(t) := \begin{pmatrix} \cos(2\lambda t) & \sin(2\lambda t) \\ \sin(2\lambda t) & -\cos(2\lambda t) \end{pmatrix}, \quad (14)$$

is a rotation matrix. We notice that Eq. (13) is a linear time-variant system. By introducing transformed coordinates $\bar{\boldsymbol{X}}_i(t) := \boldsymbol{S}(t)\boldsymbol{X}_i(t)$,

$$\boldsymbol{S}(t) := \begin{pmatrix} \cos(\lambda t) & \sin(\lambda t) \\ -\sin(\lambda t) & \cos(\lambda t) \end{pmatrix}, \quad (15)$$

we obtain the following time-invariant system:

$$\dot{\bar{\boldsymbol{X}}}(t) = [\boldsymbol{I}_5 \otimes \boldsymbol{J}_A - a^2\boldsymbol{V}^2 \otimes \boldsymbol{J}_B]\bar{\boldsymbol{X}}(t) + k(\boldsymbol{A} \otimes \boldsymbol{I}_2)\bar{\boldsymbol{X}}(t-\tau). \quad (16)$$

where $\bar{\boldsymbol{X}} := [\bar{\boldsymbol{X}}_1^T(t), \ldots, \bar{\boldsymbol{X}}_5^T(t)]^T$, $\boldsymbol{V} := \mathrm{diag}(v_1, \ldots, v_5)$,

$$\boldsymbol{J}_A := \begin{pmatrix} \mu - k & -\omega \\ \omega & \mu - k \end{pmatrix}, \quad \boldsymbol{J}_B := \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}.$$

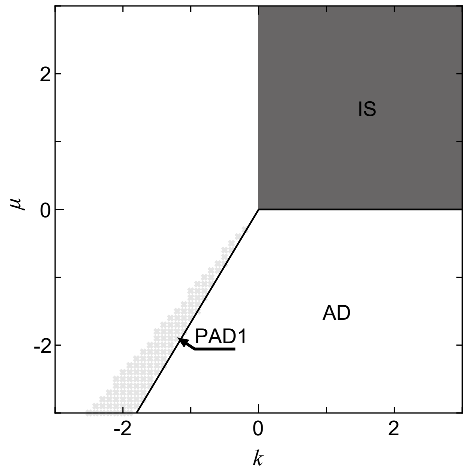
Partial amplitude death is stable if system (16) is stable. In order to estimate the stability of system (16), the present study uses DDE-BIFTOOL in Matlab.

4. Numerical examples

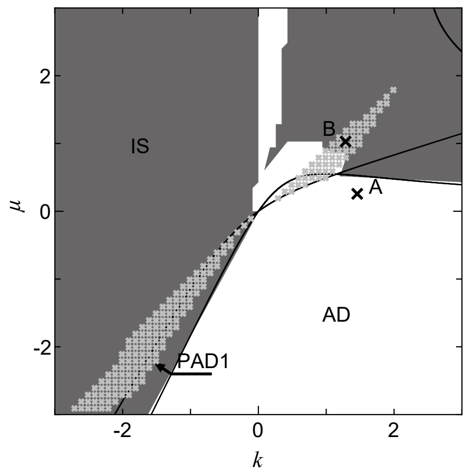
Let us estimate the stability region for partial amplitude death. The natural frequency of the oscillator is fixed at $\omega = 2$ throughout the present study.

Figure 3 shows the stability region of partial amplitude death on $(k - \mu)$ plane for the coupling delay $\tau = 0$ (i.e., static connection) and $\tau = 1.0$. The light gray area (PAD1) represents the stability region of partial amplitude death illustrated in Fig. 2(b). The stability regions for the other patterns (i.e., PAD2 and PAD3) cannot be observed. The dark gray area (IS) denotes the stability region for in-phase synchronization illustrated in Fig. 2 (a). The white area (AD) at lower right side is the stability region for amplitude death, which is derive by the procedure in [8].

For static connection (i.e., $\tau = 0$), we cannot induce partial amplitude death if the independent oscillator has the limit cycle (i.e., $\mu > 0$) [7]. However, for the delay connection (i.e., $\tau = 1.0$), we can induce partial amplitude death even if the independent oscillator has the limit cycle (i.e., $\mu > 0$). It should be noted that the previous studies [6, 7] reported that, for inducing partial amplitude death, one of the following conditions are required: the oscillators have different frequency each other; the oscillators are stabilized at steady state before coupling (i.e., $\mu < 0$).



(a) $\tau = 0$ (static connection)

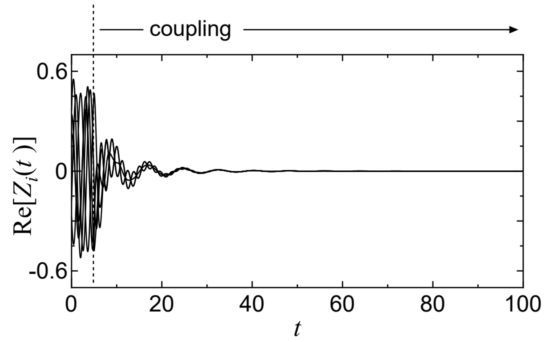


(b) $\tau = 1.0$

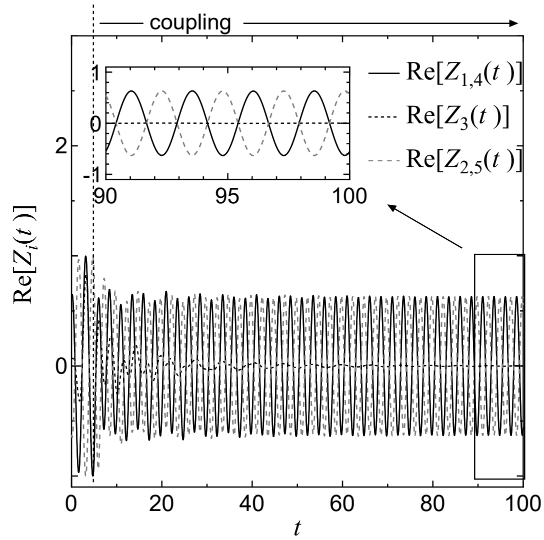
Figure 3: Stability region for partial amplitude death (PAD1) illustrated in Fig. 2(b), in-phase synchronization (IS) illustrated in Fig. 2(a), and amplitude death (AD).

In Fig. 3(b), the stability region for IS overlaps that for PAD1, that is, IS and PAD1 are bistable in the overlapped area.

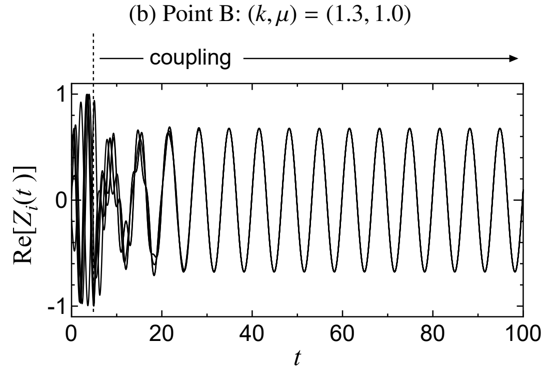
Figure 4(a) shows the time-series data of the variables at points A: $(k, \mu) = (1.5, 0.25)$ in Fig. 3(b). All the oscillators are coupled at $t = 5$. After coupling, all the variables converge onto the steady state (i.e., amplitude death). Figure 4(b) shows the time-series data at points B: $(k, \mu) = (1.3, 1.0)$ in Fig. 3(b). After coupling, we can see partial amplitude death (PAD1), that is, the 3-rd oscillator is stabilized at the origin but the others still oscillate. Figure 4(c) shows the time-series data with the same parameters used in Fig. 4(b), but with the different initial condition. Interestingly, in-phase synchronization is observed. This is because PAD1 and IS are bistable at Point B.



(a) Point A: $(k, \mu) = (1.5, 0.25)$



(b) Point B: $(k, \mu) = (1.3, 1.0)$



(c) Point B: $(k, \mu) = (1, 3.1.0)$

Figure 4: Time series data at points A and B in Fig. 3(b). We use different initial conditions for (b) and (c).

5. Conclusion

The present study investigated partial amplitude death in delay-coupled five Stuart-Landau oscillators. The solution of partial amplitude death was derived from the eigenvalues of the adjacency matrix. It was shown that the delay connection induces partial amplitude death. The analytical results were confirmed numerically.

A. The periodic solution in Eq. (10)

We assume that Eq. (10) has the periodic solution $z_\eta(t) = ae^{j\lambda t}$. Substituting it into Eq. (10) yields

$$a = \sqrt{\mu - k + k\eta \cos \lambda\tau}, \quad (17)$$

$$\lambda = \omega - k\eta \sin \lambda\tau. \quad (18)$$

By solving (18) numerically, we obtain the freqency λ. Furthermore, we can obtain amplitude a from Eq. (17). Note that Eq. (18) would have several solutions depending on τ and k.

Acknowledgments

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