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A Numerical Method for Designing Periodic Orbits Embedded in Chaotic Attractors

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Abstract—This paper proposes a framework for numerical design of continuous-time dynamical systems that bind desired periodic orbits into a chaotic attractor, with the aim of developing flexible pattern generators and controllers that can exploit various nonlinear phenomena. Our strategy is comprised of the following three stages: constructing an interim ”chaos-generating template ”, deforming the template according to the desired configuration of periodic orbits, and performing appropriate function approximation to obtain the dynamical system. In this paper, we focus on the deformations of the template, and present several numerical examples.

1. Introduction

The progress of understanding the richness of nonlinear dynamics has stimulated investigations on its applications to intelligent and flexible systems in many fields including neurocomputing, communications technology, computer vision, and robotics. Among various nonlinear phenomena such as the generation of periodic motions, their bifurcations, and synchronizations, chaos has attracted strong interests particularly from the viewpoints of complex and adaptive behaviors.

An important aspect of chaos is that chaotic attractors embed an infinite number of unstable periodic orbits (UPO's) bifurcated from pre-chaotic states [1]. Among them, some distinctive orbits can be used for characterization or control purposes. For example, a variety of chaos control methods [2, 3, 4] can stabilize UPO's embedded in chaotic attractors, enlarging the operation range and/or enhancing the functionality of the system.

In this paper, we consider such chaotic attractors as a container of UPO's (patterns) where they can be stabilized, entrained, or targeted by external inputs into the dynamical system. In particular, we propose a framework for numerical design of continuous-time dynamical systems (in the form of differential equations) that bind desirably configured UPO's into a chaotic attractor governed by a vector field (flow). Our strategy is comprised of the following three stages: constructing an interim "chaos-generating template", deforming the template according to the desired configuration of periodic orbits, and performing appropriate function approximation to obtain the dynamical system.

2. Design Aspects of Chaotic Systems

The synthesis of chaos from various approaches [5, 6, 7] has for some time been an active direction of research along the line of exploiting chaos. Primary concerns of these efforts include statistical and topological characteristics (e.g., invariant measure, Lyapunov spectrum, novel scrolling behaviors) that would be important in designing chaos-based information processing and communication applications.

In the present study, on the other hand, while sharing some common motivation with the studies mentioned above, we have put more focus on the geometrical shape and dynamical properties of UPO's themselves from the viewpoint of the adaptive generation of periodic behaviors. Here our intention lies in extending the functionality of (stable) periodic pattern generators based on function approximation of vector fields, e.g., polynomial approximation (with an application to robotics) [8] and neural network learning [9]. The present paper extends our previous proposal [10] focusing especially on the configuration of UPO's.

3. Chaos-Generating Templates

The starting point of the present study is to try to replace the periodic attractors of the polynomial vector field discussed in Ref. [8] with chaotic attractors containing a set of desired UPO's. We here consider dynamical systems of the form $\dot{x} = f(x)(x \in R^n)$, where the vector field $f(x)$ is represented by some function approximator, e.g., polynomials (as in Ref. [10]) or layered neural networks (as in the present study).

3.1. Embedding UPO's According to Chaos-Generating Mechanisms

When we consider accommodating multiple UPO's that largely overlap in the state space of a single dynamical system, haphazard approaches to placing unstable and stable manifolds can easily fail. For example, unintended stable periodic orbits may emerge from the conflict of desired instabilities, which leads to the loss of transitivity

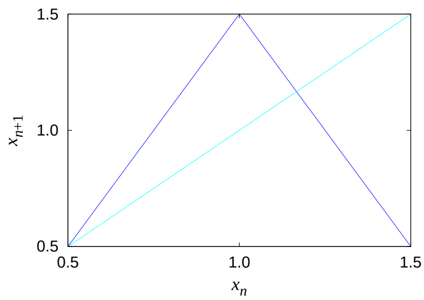

Figure 1: Nominal one-dimensional map.

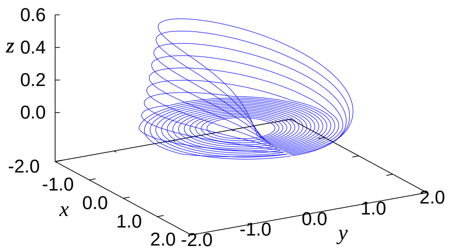

Figure 2: Nominal continuous-time chaotic flow corresponding to the map in Fig. 1.

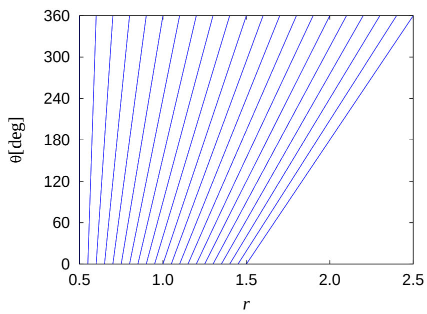

Figure 3: The stretching characteristics of the nominal chaotic flow.

among the orbits. Therefore we here consider binding UPO's according to a typical chaos-generating mechanism of stretching and folding.

First, as a simplest example, we consider the one-dimensional map (Fig. 1) for the x-coordinate of the n-th crossing on the Poincaré section $\Sigma = \{x, y, z | (y = 0, x \geq 0)\}$. While we have a wide freedom of choice of continuous-time trajectories leading to this map, we here adopt the systematically designed (explained below) bundle of orbits shown in Fig. 2 as a nominal chaotic flow for the functional approximation. This flow embeds a single period-1 UPO corresponding to the fixed point of the map.

The procedure for constructing the nominal flow in Fig. 2 is as follows: We first draw the set of orbits on the polar-coordinate $r\theta$ plane as shown in Fig. 3. In this figure, the individual trajectories starting from $(r, \theta) = (r_0, 0°)$ is

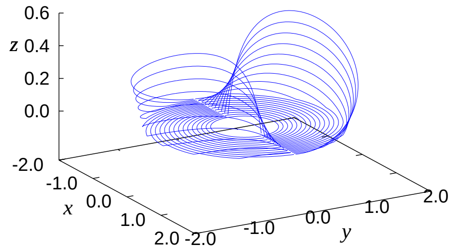

Figure 4: Nominal chaotic flow (chaos-generating template) embedding three period-1 UPO's.

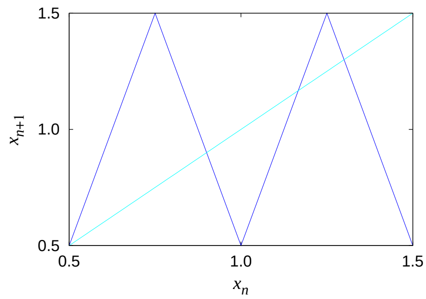

Figure 5: One-dimensional map corresponding to Fig. 4.

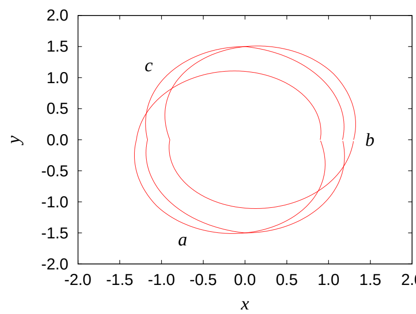

Figure 6: Embedded three period-1 UPO's.

given by

$$r = 0.5 + (r_0 - 0.5)\left(1 + \frac{\theta}{360°}\right) \tag{1}$$

$(z = 0)$, and the stretching between neighboring trajectories takes place at the rate of twice per rotation. Next, in the region where r becomes greater than 1.5, we reassign the values of r and z as

$$\begin{cases} r = 1.5 + \left[(r_0 - 0.5)\left(1 + \dfrac{\theta}{360°}\right) - 1.0\right] \cos\dfrac{\theta}{2}, \\ z = \left[(r_0 - 0.5)\left(1 + \dfrac{\theta}{360°}\right) - 1.0\right] \sin\dfrac{\theta}{2}, \end{cases} \tag{2}$$

giving the characteristic of one folding per rotation. Finally, transforming into the Cartesian coordinates, we obtain the nominal chaotic flow shown in Fig. 2.

The above strategy for embedding a single period-1 UPO can be extended to embed, or bind, several period-1 UPO's. For example, if we reorganize, or reduce, two rotations along the above nominal flow into one rotation, we obtain another nominal chaotic flow shown in Fig. 4. From the corresponding one-dimensional map in Fig. 5, we see that the flow now embeds three period-1 UPO's whose continuous-time trajectories are shown in Fig. 6. Incidentally, according to Fig. 5, the orbit starting from $(x, y, z) = (0.5, 0, 0)$ is also periodic. However, we will omit this orbit from the present study because this orbit can easily be separated from the resulting chaotic attractors.

Now we will use the above flow as a template, hereafter referred to as the chaos-generating template or simply the template, for designing a set of three coexisting period-1 UPO's.

3.2. Setting Design Points and Assigning Attracting Properties

To construct a dynamical system $\dot{x} = f(x)$ that generates a flow along the chaos-generating template shown in Fig. 4, we need to perform some function approximation such as the backpropagation learning of layered neural networks. Thus we set up design points on and in the vicinity of the template in the following manner.

First, as design points on the template, we choose 7200 points (x_i, y_i, z_i) corresponding to evenly placed 360 points on each line in Fig. 3, and specify the target velocity vector $(\dot{x}_i, \dot{y}_i, \dot{z}_i)$ along the template.

While we intend to use UPO's instead of stable periodic orbits to enhance functionality of the embedded orbits, it is desirable that the chaotic flow that binds the UPO's should be of attracting type instead of repelling or other nonattracting types. Thus we assign attracting properties by setting appropriate velocity vectors in the vicinity of the template. This can be done in various ways, and we here propose a method that we consider to be relatively easy to handle because of the small number of design parameters. As an implementation of this method, we place two additional design points $(x_i, y_i, z_i + 0.01)$ and $(x_i, y_i, z_i - 0.01)$ for each design point on the template, and assign target velocity vectors $(\dot{x}_i, \dot{y}_i, \dot{z}_i - 0.1v)$ and $(\dot{x}_i, \dot{y}_i, \dot{z}_i + 0.1v)$, respectively, where $v = \sqrt{(\dot{x}_i)^2 + (\dot{y}_i)^2 + (\dot{z}_i)^2}$. The additional velocity components define the degree of transverse stability of the flow along the template.

3.3. Embedding the Template into the Four-Dimensional State Space

The three period-1 UPO's discussed so far are entangled one another as shown in Fig. 6, which poses restrictions on the configuration of the deformed UPO's. Therefore, we here consider introducing another, fourth dimension for the state space and embedding the template into this four-dimensional ($xyzw$) space.

Here we propose an implementation in which we set design points $(x_i, y_i, z_i, w_i = x_i)$ corresponding to the design points in the previous three-dimensional space. Note that some attracting properties need to be specified before function approximation, and this can be done in a similar manner as described for the three-dimensional case.

3.4. Configuration of Unstable Periodic Orbits

In order to utilize the UPO's for specific, pattern generation applications, we need to deform the UPO's embedded in the templates according to the desired dynamical patterns. This deformation can be performed either before or after the functional approximation depending on the overall implementation. In both cases, the deformation needs to be topology-preserving to guarantee the existence of an inverse deformation that is necessary for maintaining the chaotic dynamics and the feedback path.

In the present implementation, we deform the template using a mass-spring model. Here, we treat the above-mentioned design points as mass points that are connected to eight neighboring mass points with springs. During the deformation, only the positions of the design points on the UPO's are specified, and the positions of all the other points are computed according to the energy minimization principle.

4. Numerical Example

As an example, we consider the two periodic orbits shown in Fig. 7 as the design target. Figure 8 shows the chaos-generating template (projection onto the three-dimensional space) that have been deformed so that the target periodic orbits shown in Fig. 7 are embedded in the deformed template. The pairs of the design points and the target velocity vectors on and in the vicinity of this deformed template is then used as the training set for the backpropagation learning of a three-layer neural network with four input nodes and four output nodes.

Figure 9 shows the chaotic attractor of the obtained neural-network based dynamical system. Further, Fig. 10 shows the comparison of the target periodic orbits and the periodic orbits that have been realized in the chaotic attractor and extracted by the delayed feedback. For both targets, the obtained UPO (red) is in good agreement with the desired one (blue).

5. Conclusion

We have proposed a framework for the numerical design of chaotic vector field with desirably configured periodic orbits, and demonstrated some implementations where we have successfully embedded target periodic orbits while keeping them unstable but confined in an attractor. For further discussions on the operational flexibility of the constructed systems, it will be interesting to consider other possible choices of the underlying structure of chaotic attractors [11].

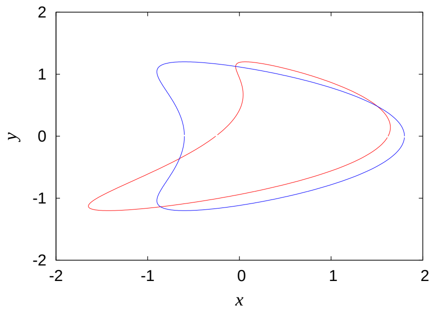

Figure 7: Two target periodic orbits.

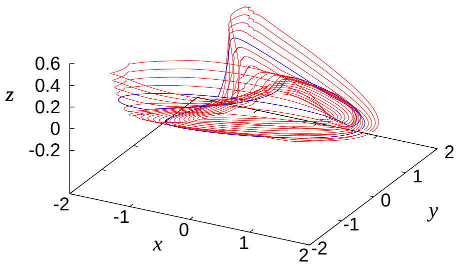

Figure 8: Deformed template that embeds the target periodic orbits.

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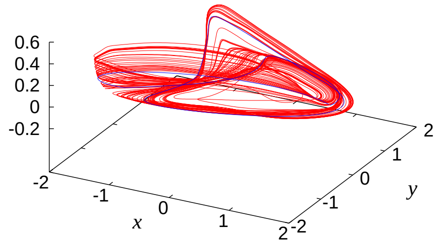

Figure 9: Chaotic attractor of the obtained neural-network based dynamical system.

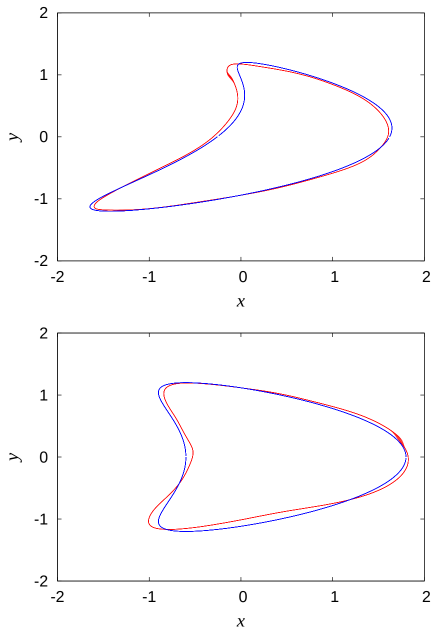

Figure 10: Comparison of the target periodic orbits (blue) and the obtained periodic orbits (red).

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