2017 International Symposium on Nonlinear Theory and Its Applications,
NOLTA2017, Cancun, Mexico, December 4-7, 2017

On Stability of State-Dependent Homogeneous Systems

Kenta Hoshino†

†Department of Electronics and Electrical Engineering, Aoyama Gakuin University
5–10–1 Fuchinobe, Sagamihara, Kanagawa 252-5258, Japan
Email: hoshino@ee.aoyama.ac.jp

Abstract—This study deals with the stability analysis of dynamical systems whose vector fields possess the homogeneity. In particular, this study focuses on the case where the homogeneity of the vector fields depends on states of dynamical systems. Although the state-dependent homogeneity has been employed in recent studies of the digital implementation of nonlinear control systems, their stability property has not been investigated. In this paper, we show that the local stability of the state-dependent homogeneous systems implies the global stability. Also, we show the existence of global homogeneous Lyapunov functions.

1. Introduction

A fundamental property of dynamical systems is stability. Furthermore, one of the fundamental problems in control theory is the stabilization problem where we design feedback controllers that guarantee the stability of closed-loop systems. One of the properties strongly related to the stability of nonlinear systems is the homogeneity.

The homogeneity is a property involving vector fields of dynamical systems under a scaling transformation (see Section 2 for definitions). The homogeneity is employed to study fundamental properties of nonlinear systems, such as not only the stability [1, 2], but also the controllability and stabilizability of nonlinear control systems [3], and properties of ordinary differential equations [4]. A notable property of the homogeneous systems is that their local asymptotic stability implies the global asymptotic stability. This property brings benefits in the stabilization of nonlinear systems because we can guarantee the global stability based on the homogeneity [5, 6].

Recently, a study [7] introduces the state-dependent homogeneity that is a generalization of the homogeneity. The study introduces the state-dependent homogeneity to develop a digital implementation of nonlinear feedback controllers. The state-dependent homogeneity can be expected to be useful for other control problems, such as the stabilization of nonlinear control systems. However, the stability of the state-dependent homogeneous systems has not been investigated. Thus, it is still unclear whether the local stability of state-dependent homogeneous systems implies the global stability.

This study provides the stability analysis of the state-dependent homogeneous systems. We show that the local stability of the state-dependent homogeneous systems implies the global stability. Moreover, we show the existence of global homogeneous Lyapunov functions, which is an extension of the result by Rosier [2]. The existence of the global homogeneous Lyapunov functions is important in control problems. The final goal of this study is to develop a design method of feedback controllers based on the state-dependent homogeneity because the state-dependent homogeneity can be amenable to a wider class of nonlinear systems. The stability analysis presented in this paper will be the basis to develop further stabilization methods.

This paper is constructed as follows. We first provide mathematical preliminaries such that the stability, and the homogeneity in the next section. Then, we provide the problem statement. In the following section, we show that the local stability of the state-dependent homogeneous systems implies the global stability. We also provide an example of the stability analysis. Finally, we give the conclusions.

2. Mathematical Preliminaries

This section introduces the definitions and results on the stability and homogeneity. Throughout this paper, we use the following notations. The notation $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{R}^n$ is the n-dimensional Euclidean space. The Lie product of vector fields f and g is denoted as $[f, g]$ which is given as $[f, g](x) = \frac{\partial g}{\partial x}(x)f(x) - \frac{\partial f}{\partial x}(x)g(x)$ for $x \in \mathbb{R}^n$.

2.1. Stability

In this paper, we discuss the stability of a dynamical system given by an ordinary differential equation

$$\dot{x} = f(x), \quad x(0) = x_0 \tag{1}$$

where $x \in \mathbb{R}^n$ is the state variable, $f : \mathbb{R}^n \to \mathbb{R}^n$, and $x_0 \in \mathbb{R}^n$ is the initial value of the state. Throughout this paper, we assume that f is a smooth vector field for the sake of simplicity. In addition, we assume that $f(0) = 0$ and that the origin $x = 0$ is a unique equilibrium of the system (1). We denote the solution to the differential equation (1) and also the flow of the vector field f as $\psi(t, x_0)$. The notation $\psi_t(\cdot)$ is also used to express $\psi(t, \cdot)$.

We first introduce the definitions of stability and asymptotic stability of the system (1).

Definition 1 (Stability)**.** The origin of the system (1) is said to be stable if for any open neighborhood U_ϵ of the origin, there exists an open neighborhood U_δ of the origin such that for any $x_0 \in U_\delta$,

$$\psi(t, x_0) \in U_\epsilon \text{ for } t \geq 0. \tag{2}$$

Definition 2 (Asymptotic Stability)**.** The origin of the system (1) is said to be locally asymptotically stable if the origin is stable and there exists an open neighborhood $U_{\delta'}$ of the origin such that for any $x_0 \in U_{\delta'}$,

$$\lim_{t\to 0} \psi(t, x_0) = 0. \tag{3}$$

In the case of $U_{\delta'} = \mathbb{R}^n$, the origin is said to be globally asymptotically stable.

Then, we introduce the Lyapunov stability theory to show the main results of this study. The Lyapunov functions are defined as follows.

Definition 3 (Lyapunov Function)**.** A function $V : U \subset \mathbb{R}^n \to \mathbb{R}$ is said to be a local Lyapunov function of the system (1) if the function $V(x)$ is continuously differentiable at $x \in U \setminus \{0\}$, positive definite (i.e., $V(0) = 0$ and $V(x) > 0$ for any $x \in U \setminus \{0\}$), proper (i.e., for any $c > 0$, the set $\{x \in U \mid V(x) \leq c\}$ is bounded), and

$$\dot{V}(x) := \frac{\partial V}{\partial x}(x) f(x) < 0 \tag{4}$$

holds for any $x \in U \setminus \{0\}$. If $U = \mathbb{R}^n$, we call $V(x)$ a global Lyapunov function of the system (1).

Then, the Lyapunov theorem is stated as follows.

Theorem 1 ([1])**.** *If a (global) Lyapunov function of the system (1) exists, then the origin of the system (1) is (globally, respectively) asymptotically stable.*

2.2. Homogeneity

This subsection introduces the homogeneity. In this paper, we consider the state-dependent homogeneity of vector fields [7].

To introduce the homogeneity, as in [8], we consider a Euler vector field given in the form of

$$\nu(x) = (r_1 x_1, r_2 x_2, \ldots, r_n x_n)^T, \tag{5}$$

where $r_i \in (0, +\infty)$ for $i = 1, \ldots, n$. The flow of the vector field $\nu(x)$ of (5) is given as

$$\phi_s(x) = (e^{r_1 s} x_1, \ldots, e^{r_n s} x_n)^T. \tag{6}$$

To introduce homogeneous functions, we consider a dilation mapping $\Delta_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ with the flow ϕ_s of (6), which is given by

$$\Delta_\lambda(x) := \phi_{\ln(\lambda)}(x) = (\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n)^T, \ \lambda \in (0, +\infty). \tag{7}$$

The flow ϕ_s of (6) given by the Euler vector field (5) and the dilation mapping Δ_λ can be viewed as a scaling mapping on the state space with respect to the parameter s and λ, respectively.

We introduce the definition of homogeneous functions.

Definition 4 (Homogeneous Function, [8])**.** A function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be a homogeneous function of degree m with respect to the Euler vector field ν of (5) if the function $V(x)$ satisfies that

$$V(\Delta_\lambda(x)) = \lambda^m V(x), \text{ for any } \lambda \in (0, +\infty), \tag{8}$$

where Δ_λ is given by (7).

Then, we introduce state-dependent homogeneous vector fields.

Definition 5 (State-Dependent Homogeneous Vector Field, [7])**.** A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be a state-dependent homogeneous vector field with a degree function $\xi : \mathbb{R}^n \to \mathbb{R}$ with respect to the Euler vector field ν of (5) if it satisfies that

$$[\nu, f](x) = \xi(x) f(x) \text{ for } x \in \mathbb{R}^n. \tag{9}$$

An example of the state-dependent homogeneous vector field is given in Example 1 of Section 4. In [7], the following result is shown.

Theorem 2 ([7])**.** *Let ϕ_s and ψ_t be the flow of the Euler vector field ν of (5) and the flow of a state-dependent homogeneous vector field f with a degree function ξ with respect to the Euler vector field ν, respectively. Then, the following relation holds:*

$$\psi_t \circ \phi_s(x) = \phi_s \circ \psi_{e^{\rho(s)} t}(x) \ \text{ where } \ \rho(s) = \int_0^s \xi \circ \phi_\tau d\tau, \ t, s \in \mathbb{R}. \tag{10}$$

Theorem 2 shows that the flow of the homogeneous vector field f scaled by the flow of the Euler vector field ν of (6), which is given by the right-hand side of (10), is also the flow of the vector field f with the scaled initial value $\phi_s(x)$, which is given by the left-hand side of (10).

In the following, we will often omit the modifier "with respect to the Euler vector fields ν" if no confusions arise. We say the system is state-dependent homogeneous if its vector field is state-dependent homogeneous.

In the rest of this paper, we will investigate the stability of state-dependent homogeneous systems.

3. Problem Statement

This section gives the problem statement of this study.

We consider the system (1) and we assume that its vector field f of the system (1) is state-dependent homogeneous with a degree function ξ with respect to the Euler vector field of (5) in the rest of this paper. Under these conditions, we show the stability analysis of the state dependent homogeneous system (1). In particular, we investigate whether the local asymptotic stability of the state-dependent homogeneous system (1) implies the global asymptotic stability.

In the case where the degree function ξ is constant, it has been already shown that the local asymptotic stability of the homogeneous system implies the global asymptotic stability. To clarify the contribution of this paper, we introduce the result in [2].

Theorem 3 ([2])**.** *Assume that the system (1) is state-dependent homogoeneous and assume that the degree function ξ of the homogeneous vector field f is constant. Also, assume that the origin of the system (1) is locally asymptotically stable. Then, the origin is globally asymptotically stable. Moreover, there exists a global homogeneous Lyapunov function of the system (1).*

The existence of the global homogeneous Lyapunov functions is important to investigate to the robust stability [2] and the analysis of the convergence rates [9].

We will consider the case where degree functions of the state homogeneous vector fields are not constant, that is, the function ξ depends on the states x of the systems. We will show a generalization of Theorem 3 in the next section.

4. Main Results: Stability of State-Dependent Homogeneous Systems

This section presents a stability analysis of the state-dependent homogeneous systems. We will show that the local stability of state-dependent homogeneous systems implies the global stability.

The following theorem is the main result of this study.

Theorem 4. *Assume that the vector field f of the system (1) is homogeneous with a degree function ξ with respect to the Euler vector fields v of (5). Further, we assume that the origin of the system (1) is locally asymptotically stable. Then, the origin of the system (1) is globally asymptotically stable. Moreover, there exists a global homogeneous Lyapunov function of the system (1).*

Proof. Due to the limited space, we only give an outline of the proof. We show the global asymptotic stability of the system (1) from the local stability as with the proof in [2] by using Theorem 2.

Because the origin of the system (1) is locally stable, there exists an open neighborhood $U_{\delta'}$ of the origin such that (3) holds. Then, according to (6), we can easily show that for any $x_0 \in \mathbb{R}^n \setminus \{0\}$, there exists $s' \in \mathbb{R}$ such that $\phi_{s'}(x_0) \in U_{\delta'}$. By using s', we consider a solution to (1) with the initial value $\phi_{s'}(x_0)$, that is, $\bar{\psi}_t(x_0) := \psi_t \circ \phi_{s'}(x_0)$. Then, because $\phi_{s'}(x_0) \in U_{\delta'}$, the local asymptotic stability implies that

$$\lim_{t\to\infty} \bar{\psi}_t(x_0) = 0 \tag{11}$$

holds. Then, we consider $\bar{\bar{\psi}}_t(x_0) = \phi_{-s'} \circ \bar{\psi}_{e^{-\rho(s')}t}(x_0)$ where $\rho(\cdot)$ is given in (10). Because the map $\phi_s(x)$ of (6) induces a scaling mapping with respect to the state variable and $e^{-\rho(s')} > 0$, we obtain from (11) that

$$\lim_{t\to\infty} \bar{\bar{\psi}}_t(x_0) = 0. \tag{12}$$

However, by using $\bar{\psi}_{e^{-\rho(s')}t}(x_0) = \psi_{e^{-\rho(s')}t} \circ \phi_{s'}(x_0)$ and Theorem 2, we obtain that

$$\begin{aligned} \bar{\bar{\psi}}_t(x_0) &= \phi_{-s'} \circ \bar{\psi}_{e^{-\rho(s')}t}(x_0) = \phi_{-s'} \circ \psi_{e^{-\rho(s')}t} \circ \phi_{s'}(x_0) \\ &= \phi_{-s'} \circ \phi_{s'} \circ \psi_{e^{\rho(s')}\cdot(e^{-\rho(s')}t)}(x_0) = \psi_t(x_0). \end{aligned} \tag{13}$$

By (12) and (13), we can conclude that

$$\lim_{t\to\infty} \psi_t(x_0) = 0, \tag{14}$$

where ψ_t is a solution to (1) with the initial value $x_0 \in \mathbb{R}^n \setminus \{0\}$. This implies the global asymptotic stability of the system (1).

Then, we show the existence of a homogeneous global Lyapunov function of the system (1). Because we have shown the global stability of the system (1), according to the converse Lyapunov theorem by Kurzweil [10], there exists a global Lyapunov function $V(x)$ of the system (1), which is not necessarily homogeneous. Then, as done in [2], we construct a homogeneous Lyapunov function candidate $\bar{V}(x)$ given by

$$\bar{V}(x) := \begin{cases} \int_0^{+\infty} \frac{1}{\lambda^{k+1}} (a \circ V \circ \Delta_\lambda)(x) d\lambda, & \forall x \in \mathbb{R}^n \setminus \{0\} \\ 0, & x = 0 \end{cases} \tag{15}$$

where k is a positive integer, Δ_λ is a dilation mapping given in (7), and the function $a : \mathbb{R} \to \mathbb{R}$ is a smooth function such that

$$a(s) = \begin{cases} 0 & \text{for } s \in (-\infty, 1] \\ 1 & \text{for } s \in [2, \infty) \end{cases}, \quad a'(s) \geq 0 \text{ for } \forall s \in \mathbb{R}. \tag{16}$$

We can show that this function $\bar{V}(x)$ of (15) is homogeneous with respect to the Euler vector field v of (5), positive definite, and proper.

Moreover, we can show that

$$f(\Delta_\lambda(x)) = e^{\rho(\ln(\lambda))} \Delta_\lambda(f(x)), \ \lambda \in (0, +\infty) \tag{17}$$

holds for the state-dependent homogeneous vector field f, where $\rho(s)$ is given in (10), and Δ_λ is given by (7). By using the equation (17), we can show that

$$\dot{\bar{V}}(x) := \frac{\partial \bar{V}}{\partial x}(x) f(x) < 0 \tag{18}$$

holds for any $x \in \mathbb{R}^n \setminus \{0\}$. Then, according to Definition 3 and Theorem 1, this indicates that the function $\bar{V}(x)$ of (15) is a homogeneous global Lyapunov function of the system (1).

□

We show a simple example of the state-dependent homogeneous systems and the stability analysis based on Theorem 4. The following example is taken from [7] where the example is considered for the self-triggered control, and its stability is not investigated in [7].

Example 1. Let us consider a system given by

$$\dot{x} = f(x), \quad \text{where} \quad f(x) = \left[-x_1 - x_1^3, -x_2 - x_1^2 x_2\right]^T. \qquad (19)$$

For the vector field f of (19), we consider the Euler vector field given as

$$\nu(x) = [x_1, x_2]^T, \qquad (20)$$

which is obtained by setting $r_1 = r_2 = 1$ in (5). Then, we obtain that

$$[\nu, f](x) = \frac{2x_1^2}{x_1^2 + 1} f(x), \qquad (21)$$

and the equation (21) shows that the vector field $f(x)$ of (19) is a state-dependent homogeneous vector field with the degree function $\xi(x) = (2x_1^2)/(x_1^2 + 1)$ with respect to the Euler vector field ν of (20).

Then, by investigating a linearized system of the system (19), we can easily find that the system (19) is locally asymptotically stable. Thus, because the vector field $f(x)$ given in (19) is state-dependent homogeneous, and the origin is locally asymptotically stable, we can conclude the global asymptotic stability of the system from Theorem 4. Indeed, we can easily find the global Lyapunov function of the system (19), for example, $V(x) = x_1^2 + x_2^2$.

Figure 1 shows an example of the time response of the state $x(t)$ of the system (19) with the initial value $(x_1(0), x_2(0)) = (1, -1)$. Because of the global asymtptotic stability of (19), the state x of the system (19) converges to the origin with any initial value of the state $x_0 \in \mathbb{R}^2$.

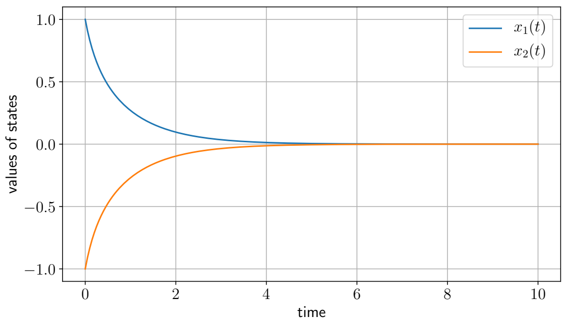

Figure 1: Example of trajectory $x(t)$ of the system (19)

5. Conclusions

In this paper, we presented the stability analysis of the state-dependent homogeneous systems. As the main result, this study shows that the local stability of the state-dependent homogeneous systems implies the global stability of the systems. This is different from the case of general nonlinear systems for which the local stability does not imply the global stability. In this paper, we just provided the stability analysis of state-dependent homogeneous systems. In future works, we will deal with the stabilization of nonlinear control systems based on the state-dependent homogeneity. We can expect the global stabilization by considering the state-dependent homogeneity. Moreover, the state-independent homogeneity plays an important role in the analysis and synthesis of the convergence rates of states, which include the finite-time stability that is an active topic in nonlinear control theory [1, 9]. We will investigate the convergence rates of the state-dependent homogeneous systems in future work.

References

[1] A. Bacciotti and L. Rosier, *Liapunov functions and stability in control theory*. Springer-Verlag Berlin Heidelberg, 2005.

[2] L. Rosier, "Homogeneous Lyapunov function for homogeneous continuous vector field," *Systems & Control Letters*, vol. 19, no. 6, pp. 467–473, 1992.

[3] H. Hermes, "Nilpotent and high-order approximations of vector field systems," *SIAM review*, vol. 33, no. 2, pp. 238–264, 1991.

[4] V. I. Arnold, *Ordinary Differential Equations*. Springer-Verlag Berlin Heidelberg New York, 1992.

[5] L. Grüne, "Homogeneous state feedback stabilization of homogenous systems," *SIAM Journal on Control and Optimization*, vol. 38, no. 4, pp. 1288–1308, 2000.

[6] N. Nakamura, H. Nakamura, and H. Nishitani, "Global inverse optimal control with guaranteed convergence rates of input affine nonlinear systems," *Automatic Control, IEEE Transactions on*, vol. 56, no. 2, pp. 358–369, 2011.

[7] A. Anta and P. Tabuada, "To sample or not to sample: Self-triggered control for nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 9, pp. 2030–2042, 2010.

[8] M. Kawski, "Geometric homogeneity and stabilization," in *Proceedings of IFAC Nonlinear Control Systems Design Symposium (NOLCOS95)*, pp. 164–169, 1995.

[9] S. P. Bhat and D. S. Bernstein, "Finite-time stability of homogeneous systems," in *Proceedings of the 1997 American Control Conference, 1997.*, pp. 2513–2514, 1997.

[10] J. Kurzweil, "On the inversion of Ljapunov's second theorem on stability of motion," *American Mathematical Society Translations, Series 2*, no. 24, pp. 19–77, 1963.